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# A CONJECTURE OF CHU ET AL. AND A NEW FAMILY OF MSTD SETS

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**Abstract**

Let $A$ be a nonempty finite set of integers. The *sumset* and *difference set* of $A$ are defined as follows

$$A + A = \{a + b : a, b \in A\},$$
$$A - A = \{a - b : a, b \in A\}.$$

Then $A$ is said to be *sum-dominant* or *more-sum-than-difference* (MSTD) if $|A + A| > |A - A|$. Recently, Chu et al. [3] gave an infinite family $\mathcal{F}$ of finite sets and conjectured that all sets in the family were MSTD. They proved that some periodic subfamilies of this family are MSTD. In this article, we give a non-periodic subfamily that contains no MSTD sets, thus disproving the above conjecture. Finally, we generalize Chu et al.'s subfamily of MSTD sets to obtain a more general collection of MSTD sets.

## 1. Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of positive integers and all integers, respectively. For integers $a$ and $b$, let $[a, b] = \{x \in \mathbb{Z} \mid a \le x \le b\}$. Let $A$ be a nonempty finite set of integers. Define the *sumset* and the *difference set* of $A$, respectively, as

$$A + A := \{a + b : a, b \in A\},$$

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and

$$A - A := \{a - b : a, b \in A\}.$$

For an integer $x$, define

$$\{x\} - A := \{x - a : a \in A\}.$$

The cardinality of $A$ is denoted by $|A|$. Set $A$ is said to be

- *sum-dominated* (or *MSTD*) if $|A + A| > |A - A|$,
- *balanced* if $|A + A| = |A - A|$,
- *difference-dominated* if $|A + A| < |A - A|$,
- an *arithmetic progression* of length $k$ if $A = \{a + id : i \in [0, k - 1]\}$, for some $a, d \in \mathbb{Z}$,
- *symmetric* if there exists an integer $x$ such that $A = \{x\} - A$.

Let $\mathcal{X}$ be a family of sets. Then $\mathcal{X}$ is said to be a

- *sum-dominated family* (or *MSTD family*) if $|A + A| > |A - A|$ for all $A \in \mathcal{X}$,
- *balanced family* if $|A + A| = |A - A|$ for all $A \in \mathcal{X}$,
- *difference-dominated family* if $|A + A| < |A - A|$ for all $A \in \mathcal{X}$.

In an abelian group $G$, the addition of two distinct elements is commutative, that is,

$$a + b = b + a;$$

but the subtraction of two distinct elements may not be, that is, it is possible that

$$a - b \neq b - a,$$

unless $a - b$ has order 2. This certainly suggests that most finite sets $A$ satisfy $|A+A| \leq |A-A|$. In support to this, Roesler [14] proved that the average value of $|A-A|$ over the average value of $|A+A|$, where $A$ is a subset of $\{0, 1, 2, \ldots, n\}$ with $k$ elements, lies in $[1, 2)$. In the 1960s, it was an open question whether an MSTD set exists or not. On the other hand, it is easy to construct balanced sets. Every symmetric set, for example, is a balanced set.

It is believed that Conway gave the first example, $A = \{0, 2, 3, 4, 7, 11, 12, 14\}$, of an MSTD set in 1969, and after that, a lot of work has been done on MSTD sets (see [7, 8, 13, 12, 15]). Martin and O'Bryant [9] proved that the proportion of MSTD subsets of $\{0, 1, 2, \ldots, n-1\}$ is bounded below by a positive constant (about $2 \cdot 10^{-7}$) as $n \to \infty$, which later was improved by Zhao [18] to about $4 \cdot 10^{-4}$. Using the *base expansion method* [7, 12], we can construct a new MSTD set from a given

MSTD set. Nathanson [13] gave the first explicit construction of infinite families of MSTD sets. After that, many including Hegarty [7], Miller et al. [10] and Miller et al. [11] gave explicit construction of an infinite family of MSTD sets. Zhao [17] used *bidirectional ballot sequences* to construct a large family of MSTD sets.

Hegarty [7] also proved that there is no MSTD set of cardinality less than 8 using clever algorithms. Chu [2, 4] gave a different and computer-free proof that an MSTD set must have at least 7 elements. Also, Chu et al. [1] and Chu [5] gave some interesting families of sets that are not MSTD. Inspired by some already existing MSTD sets, Chu et al. [3] gave an infinite family $\mathcal{F}$ of sets and conjectured that $\mathcal{F}$ is MSTD. They showed that the conjecture holds for a periodic subfamily of $\mathcal{F}$.

In Section 2, we first give a balanced subfamily of $\mathcal{F}$ (Theorem 1) and a difference-dominated subfamily of $\mathcal{F}$ (Theorem 2), which are counter-examples to the conjecture of Chu et al. [3]. Next, we give some non-periodic MSTD subfamilies of $\mathcal{F}$ (Theorem 3 and Theorem 4). In Section 3, we give a more general family than $\mathcal{F}$, denoted by $\mathcal{T}$, and give a periodic MSTD subfamily of $\mathcal{T}$ in Theorem 5.

We use the following notation in the paper. The *positive subset* of a set $A$, denoted by $A^+$, is defined as

$$A^+ = \{a \in A : a > 0\}.$$

Therefore

$$A - A = -(A-A)^+ \cup \{0\} \cup (A-A)^+.$$

So $|A-A| = 2|(A-A)^+| + 1$. It is easy to check that

$$((\alpha \cdot A) + \beta) + ((\alpha \cdot A) + \beta) = \alpha \cdot (A + A) + 2\beta$$

and

$$((\alpha \cdot A) + \beta) - ((\alpha \cdot A) + \beta) = \alpha \cdot (A - A),$$

for some $\alpha, \beta \in \mathbb{R}$. So $|A+A|$ and $|A-A|$ are translation and dilation invariant. So, we assume that $A$ is a set of non-negative integers with $\min(A) = 0$ and $d(A) = 1$, where

$$d(A) = \gcd\{a_i : a_i \in A\}.$$

In 1973, Spohn [16] introduced a way to represent a set of integers. Let $X = \{x_1, x_2, x_3, \ldots, x_n\}$ be a finite set of integers such that $x_1 < x_2 < \cdots < x_n$. The *sequence of consecutive differences* (SCD) of set $X$ is the sequence

$$x_2 - x_1, x_3 - x_2, x_4 - x_3, \ldots, x_n - x_{n-1}.$$

Then $X = (x_1 \mid x_2 - x_1, x_3 - x_2, \ldots, x_n - x_{n-1})$. For example, if $A = \{0, 1, 6, 7, 8, 17, 24\}$, then the SCD of $A$ is $1, 5, 1, 1, 9, 7$ and $A = (0 \mid 1, 5, 1, 1, 9, 7)$. Conversely, for a given $A = (a_1 \mid a_2, \ldots, a_k)$, we can get the underlying set

$$A = \left\{ \sum_{i=1}^{t} a_i : t \in [1, k] \right\}.$$

For a given set $A = (a_1 \mid a_2, \ldots, a_n)$, by a *run* of $A$, we mean

$$a_i, a_{i+1}, \ldots, a_j,$$

where $2 \leq i \leq j \leq n$. The sum of a run gives a positive element of $A - A$; in fact,

$$(A - A)^+ = \left\{ \sum_{t=i}^{j} a_t : a_i, a_{i+1}, \ldots, a_j \in SCD \text{ with } 2 \leq i \leq j \leq n \right\}.$$

Consider the following MSTD sets, where $A_1$ and $A_2$ are found in [9], $A_3$ and $A_4$ are found in [7], and $A_5$ is found in [6];

$$\begin{aligned}
&A_1 = \{0, 1, 2, 4, 5, 9, 12, 13, 14, 16, 17\},\\
&A_2 = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25\},\\
&A_3 = \{0, 1, 2, 4, 5, 9, 12, 13, 14\},\\
&A_4 = \{0, 1, 2, 4, 5, 9, 12, 13, 17, 20, 21, 22, 24, 25, 29, 32, 33, 37, 40, 41, 42, 44, 45\},\\
&A_5 = \{0, 1, 2, 4, 5, 9, 12, 13, 14, 16, 17, 21, 24, 25, 26, 28, 29\}.
\end{aligned}$$

These sets appeared random initially, but their SCD follows a pattern. We can write $A_1$, $A_2$, and $A_3$ in their SCD form as follows:

$$\begin{aligned}
A_1 &= (0 \mid 1, 1, 2, 1, 4, 3, 1, 1, 2, 1),\\
A_2 &= (0 \mid 1, 1, 2, 1, 4, 3, 1, 4, 3, 1, 1, 2, 1),\\
A_3 &= (0 \mid 1, 1, 2, 1, 4, 3, 1, 1).
\end{aligned}$$

Inspired by the above MSTD sets, Chu et al. [3] gave an infinite family $\mathcal{F}$ of sets, which is described as follows.

Let $k_1, k_2, k_3, \ldots, k_l$ and $l$ be positive integers. Let $M^k$ denote the sequence $1, \underbrace{4, \ldots, 4}_{k \text{ times}}, 3$. Then $\mathcal{F}$ is the family of sets $A$ such that

$$A = (0 \mid 1, 1, 2, M^{k_1}, M^{k_2}, M^{k_3}, \ldots, M^{k_l}, M_1),$$

where $M_1$ is either $1, 1$ or $1, 1, 2$ or $1, 1, 2, 1$. Note that there is a typo in the definition of $\mathcal{F}$ in the paper [3]. Instead of $1, 1, 2, 1$ (in the starting of SCD) it should be $1, 1, 2$. Also, we could not achieve the set $A_{15}$ (which is $A_4$ in our paper) by the family $\mathcal{F}$. In the same article, they propose the following conjecture.

**Conjecture 1** ([3, Conjecture 1.3]). All sets in $\mathcal{F}$ are MSTD.

To support Conjecture 1, a subfamily of $\mathcal{F}$ was given by Chu at el. [3]. Conjecture 1 is not true; in fact, we have a balanced subfamily of $\mathcal{F}$ (see Theorem 1) and a difference-dominated subfamily of $\mathcal{F}$ (see Theorem 2).

Now, considering the sets $A_4$ and $A_5$, we have

$$A_4 = (0 \mid 1,1,2,1,4,3,1,4,3,1,1,2,1,4,3,1,4,3,1,1,2,1),$$
$$A_5 = (0 \mid 1,1,2,1,4,3,1,1,2,1,4,3,1,1,2,1).$$

Inspired by these examples, we provide a more general family of sets by repeating the full block of interior blocks.

**Definition 1.** Let $l, t, k_1, k_2, \ldots, k_l$ be positive integers and $M^{k_i}$ denote the sequence $1, \underbrace{4, \ldots, 4}_{k_i \text{ times}}, 3$. Let $\mathcal{T}$ be the family of sets $A$ such that

$$A = (0 \mid 1,1,2, \underbrace{\underbrace{M^{k_1}, \ldots, M^{k_l}, M_1}, \underbrace{M^{k_1}, \ldots, M^{k_l}, M_2}, \ldots, \underbrace{M^{k_1}, \ldots, M^{k_l}, M_t}}_{t \text{ times}}),$$

where $M_i$ is either $1,1$ or $1,1,2$ or $1,1,2,1$ for $i \in [1,t]$.

**Remark 1.** If $t = 1$, then $\mathcal{T} = \mathcal{F}$.

## 2. Some Balanced, Difference-Dominated, and MSTD Subfamilies of $\mathcal{T}$

**Theorem 1.** *Let $k$ be a positive integer. Let $\mathcal{M}$ be the family of sets $A$ such that*

$$A = (0 \mid 1,1,2, M^k, M^{k+1}, 1,1).$$

*Then $\mathcal{M}$ is a balanced family.*

*Proof.* We have

$$A = (0 \mid 1,1,2,1, \underbrace{4,4,\ldots,4}_{k \text{ times}}, 3,1, \underbrace{4,4,\ldots,4}_{k+1 \text{ times}}, 3,1,1).$$

Then $A$ can be written explicitly as

$$A = \{0, 2, 4, 8+4k, 16+8k, 18+8k\} \cup \{1+4i : i \in [0, 2k+4]\}.$$

Therefore, for $i \in [0, 2k+4]$, we have

$$1 + 4i = 0 + (1+4i) \in A + A,$$

$$2 + 4i = 1 + (1+4i) \in A + A,$$

and

$$3 + 4i = 2 + (1+4i) \in A + A.$$

Also, for $j \in [2k+4, 4k+8]$, we have

$$1 + 4j = (1 + 4j - 16 - 8k) + (16 + 8k) \in A + A,$$

$$2 + 4j = (2 + 4j - 17 - 8k) + (17 + 8k) \in A + A,$$

and

$$3 + 4j = (3 + 4j - 18 - 8k) + (18 + 8k) \in A + A.$$

Thus, $A + A$ contains all integers of the form $1 + 4i$, $2 + 4i$, and $3 + 4i$, where $i \in [0, 4k+8]$. Note also that the only elements of the form $4m$, in $A + A$, are

$$0, 4, 8, 8 + 4k, 12 + 4k, 16 + 8k, 20 + 8k, 24 + 12k, 32 + 16k, \text{and } 36 + 16k.$$

Hence $|A + A| = 37 + 12k$.

Next, we find $|A - A|$ by finding $(A - A)^+$. For $i \in [0, 2k+4]$, we have

$$1 + 4i \in A^+ \subseteq (A - A)^+.$$

Since the sum of a run gives an element of $(A - A)^+$, we have the following:

1. Run $1, 2, 1, \underbrace{4, 4, \ldots, 4}_{i \text{ times}}$ gives $4 + 4i \in (A - A)^+$, where $i \in [0, k]$.
2. Run $1, 2, 1, \underbrace{4, 4, \ldots, 4}_{k \text{ times}}, 3, 1, \underbrace{4, 4, \ldots, 4}_{i \text{ times}}$ gives $8 + 4k + 4i \in (A - A)^+$, where $i \in [0, k+1]$.
3. Run $1, 2, 1, \underbrace{4, 4, \ldots, 4}_{k \text{ times}}, 3, 1, \underbrace{4, 4, \ldots, 4}_{k+1 \text{ times}}, 3, 1$ gives $16 + 8k \in (A - A)^+$.

Thus, $4i \in (A - A)^+$ for all $i \in [1, 2k+4]$. Using a similar argument, we can prove by starting each run at $2, 1$, that $3 + 4i \in (A - A)^+$ for all $i \in [0, 2k+3]$. Note that

$$\begin{aligned}
2 - 0 = 2 &\in (A - A)^+, \\
8 + 4k - 2 = 6 + 4k &\in (A - A)^+, \\
18 + 8k - (8 + 4k) = 10 + 4k &\in (A - A)^+, \\
18 + 8k - 4 = 14 + 8k &\in (A - A)^+, \\
18 + 8k - 0 = 18 + 8k &\in (A - A)^+.
\end{aligned}$$

These are the only elements of the form $2 + 4m$ that are in $(A - A)^+$. Hence,

$$|(A - A)^+| = 6k + 18.$$

So

$$|A - A| = 37 + 12k.$$

Thus, $A$ is a balanced set. $\square$

We omit the proofs of the following two theorems as they are very similar to the proof of Theorem 1.

**Theorem 2.** *Let $k$ be a positive integer. Let $\mathcal{M}$ be the family of sets $A$ such that*

$$A = (0 \mid 1, 1, 2, M^k, M^{k+1}, M^{k+1}, 1, 1).$$

*Then $\mathcal{M}$ is a difference-dominated family.*

**Theorem 3.** *Let $k$ be a positive integer. Let $\mathcal{M}$ be the family of sets $A$ such that*

$$A = (0 \mid 1, 1, 2, M^k, M^{k+1}, M_1),$$

*where $M_1$ is either $1, 1, 2$ or $1, 1, 2, 1$. Then $\mathcal{M}$ is an MSTD family.*

**Theorem 4.** *Let $k$ be a positive integer. Let $\mathcal{M}$ be the family of sets $A$ such that*

$$A = (0 \mid 1, 1, 2, M^{k+1}, M^k, M_1),$$

*where $M_1$ is either $1, 1$ or $1, 1, 2$ or $1, 1, 2, 1$. Then $\mathcal{M}$ is an MSTD family.*

*Proof.* Let $M_1$ be $1, 1$. We have

$$A = (0 \mid 1, 1, 2, 1, \underbrace{4, 4, \ldots, 4}_{k+1 \text{ times}}, 3, 1, \underbrace{4, 4, \ldots, 4}_{k \text{ times}}, 3, 1, 1).$$

Then $A$ can be written explicitly as

$$A = \{0, 2, 4, 12 + 4k, 16 + 8k, 18 + 8k\} \cup \{1 + 4i : i \in [0, 2k + 4]\}.$$

Therefore, for $i \in [0, 2k + 4]$, we have

$$1 + 4i = 0 + (1 + 4i) \in A + A,$$

$$2 + 4i = 1 + (1 + 4i) \in A + A,$$

and

$$3 + 4i = 2 + (1 + 4i) \in A + A.$$

Also, for $j \in [2k + 4, 4k + 8]$, we have

$$1 + 4j = (1 + 4j - 16 - 8k) + (16 + 8k) \in A + A,$$

$$2 + 4j = (2 + 4j - 17 - 8k) + (17 + 8k) \in A + A,$$

and

$$3 + 4j = (3 + 4j - 18 - 8k) + (18 + 8k) \in A + A.$$

Thus, $A + A$ contains all integers of the form $1 + 4i$, $2 + 4i$, and $3 + 4i$, where $i \in [0, 4k + 8]$. Note also that the only elements of the form $4m$, that are in $A + A$, are

$$0, 4, 8, 12 + 4k, 16 + 4k, 16 + 8k, 20 + 8k, 24 + 8k, 28 + 12k, 32 + 16k, \text{and } 36 + 16k.$$

Hence, $|A + A| = 38 + 12k$.

Next, we find $|A - A|$ by finding $(A - A)^+$. We have, for $i \in [0, 2k + 4]$,

$$1 + 4i \in A^+ \subseteq (A - A)^+.$$

Since the sum of a run gives an element of $(A - A)^+$, we have the following:

1. Run $1, 2, 1, \underbrace{4, 4, \ldots, 4}_{i \text{ times}}$ gives $4 + 4i \in (A - A)^+$, where $i \in [0, k + 1]$.
2. Run $1, 2, 1, \underbrace{4, 4, \ldots, 4}_{k+1 \text{ times}}, 3, 1, \underbrace{4, 4, \ldots, 4}_{i \text{ times}}$ gives $12 + 4k + 4i \in (A - A)^+$, where $i \in [0, k]$.
3. Run $1, 2, 1, \underbrace{4, 4, \ldots, 4}_{k+1 \text{ times}}, 3, 1, \underbrace{4, 4, \ldots, 4}_{k \text{ times}}, 3, 1$ gives $16 + 8k \in (A - A)^+$.

Thus, $4i \in (A - A)^+$ for all $i \in [1, 2k + 4]$. Using a similar argument, we can prove by starting each run at $2, 1$, that $3 + 4i \in (A - A)^+$ for all $i \in [0, 2k + 3]$. Note that

$$\begin{aligned} 2 - 0 = 2 &\in (A - A)^+, \\ 18 + 8k - (12 + 4k) = 6 + 4k &\in (A - A)^+, \\ 12 + 4k - 2 = 10 + 4k &\in (A - A)^+, \\ 18 + 8k - 4 = 14 + 8k &\in (A - A)^+, \\ 18 + 8k - 0 = 18 + 8k &\in (A - A)^+. \end{aligned}$$

These are the only elements of the form $2 + 4m$ that are in $(A - A)^+$. Hence

$$|(A - A)^+| = 6k + 18.$$

So

$$|A - A| = 37 + 12k.$$

Thus, $A$ is an MSTD set. The proofs when $M_1$ is either $1, 1, 2$ or $1, 1, 2, 1$, are similar to the case when $M_1$ is $1, 1$, so we omit the details. $\square$

## 3. A Periodic MSTD Subfamily of $\mathcal{T}$

Let $\mathcal{W}(k,l,t)$ be a periodic subfamily of $\mathcal{T}$ in which $k_1 = k_2 = \cdots = k_l = k$, $M_i$ is $1,1,2$ for $i \in [1, t-1]$, and $M_t$ is either $1,1$ or $1,1,2$ or $1,1,2,1$. Then for any set $A \in \mathcal{W}(k,l,t)$, we have

$$A = (0 \mid \underbrace{\underbrace{1,1,2,M^k,\ldots,M^k}, \underbrace{1,1,2,M^k,\ldots,M^k}, \ldots, \underbrace{1,1,2,M^k,\ldots,M^k}}_{t \text{ times}}, M_t),$$

where the block $\underbrace{1,1,2,M^k,\ldots,M^k}$ represents

$$1,1,2,\underbrace{1,\underbrace{4,\ldots,4}_{k \text{ times}}3,1,\underbrace{4,\ldots,4}_{k \text{ times}},3,\ldots,1,\underbrace{4,\ldots,4}_{k \text{ times}},3}_{l \text{ times}}.$$

**Remark 2.** From now on, unless otherwise stated, $M^k$ is repeated $l$ times in the block $\underbrace{M^k,\ldots,M^k}$.

For $t = 1$, Chu et al. proved that $\mathcal{W}(k,l,t)$ is MSTD ([3, Theorem 1.6]). In Theorem 5, we prove $\mathcal{W}(k,l,t)$ is MSTD for all $t \geq 1$.

**Theorem 5.** *Let $k$, $l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1,1,2,M^k,\ldots,M^k}, \underbrace{1,1,2,M^k,\ldots,M^k}, \ldots, \underbrace{1,1,2,M^k,\ldots,M^k}}_{t \text{ times}} M_t),$$

*where $M^k = 1,\underbrace{4,4,\cdots,4}_{k \text{ times}},3$ and $M_t$ is either $1,1$ or $1,1,2$ or $1,1,2,1$. Then $A$ is an MSTD set.*

*Proof.* We prove the theorem when $M_t$ is $1,1,2,1$. The proofs when $M_t$ is $1,1$ or $1,1,2$ are similar. The proof is divided into two subsections. In Subsection 3.1, we prove that $|A - A| = 6klt + 8lt + 8t + 11$ and in Subsection 3.2, we prove that $|A + A| \geq 6klt + 10lt + 8t + 11$. Therefore

$$|A + A| - |A - A| \geq 6klt + 10lt + 8t + 11 - (6klt + 8lt + 8t + 11) = 2lt > 0.$$

This proves that $A$ is an MSTD set. $\square$

### 3.1. Difference Set

**Lemma 1.** *Let $k$, $l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1,1,2,M^k,\ldots,M^k}, \underbrace{1,1,2,M^k,\ldots,M^k}, \ldots, \underbrace{1,1,2,M^k,\ldots,M^k}}_{t \text{ times}}, 1,1,2,1).$$

*Then the following hold:*

(1) $4i \in (A-A)^+$, *where* $i \in [1, 1+t(kl+l+1)]$,

(2) $1+4i \in (A-A)^+$, *where* $i \in [1, 1+t(kl+l+1)]$,

(3) $3+4i \in (A-A)^+$, *where* $i \in [0, t(kl+l+1)]$.

*Proof.* Observe that the set of partial sums of the run

$$1, 2, \underbrace{\underbrace{M^k, \ldots, M^k, 1, 1, 2}, \ldots, \underbrace{M^k, \ldots, M^k, 1, 1, 2}}_{t \text{ times}}, 1$$

contains all multiples of 4 from 4 to $4 + 4t(kl+l+1)$. This proves Item (1) of the lemma. Similarly, we obtain Item (2) and Item (3) of the lemma by considering the runs

$$1, 1, 2, \underbrace{\underbrace{M^k, \ldots, M^k, 1, 1, 2}, \ldots, \underbrace{M^k, \ldots, M^k, 1, 1, 2}}_{t \text{ times}}, 1$$

and

$$2, \underbrace{\underbrace{M^k, \ldots, M^k, 1, 1, 2}, \ldots, \underbrace{M^k, \ldots, M^k, 1, 1, 2}}_{t \text{ times}}, 1,$$

respectively. This completes the proof of the lemma. □

Now we consider elements of the form $2+4i$, where $i \in \mathbb{Z}$. It is easy to check that $2 \in A - A$ because of the run $1, 1$. We find all possible runs which give the sums of the form $6+4i$, in the next lemma.

**Lemma 2.** *Let $k$, $l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1, 1, 2, M^k, \ldots, M^k}, \underbrace{1, 1, 2, M^k, \ldots, M^k}, \ldots, \underbrace{1, 1, 2, M^k, \ldots, M^k}}_{t \ \textit{times}}, 1, 1, 2, 1).$$

*Any run that generates elements of the form $6+4m$, must have one of the following forms:*

$(R_1)$

$$1, 1, \underbrace{\underbrace{2, M^k, \ldots, M^k, 1, 1}, \underbrace{2, M^k, \ldots, M^k, 1, 1}, \ldots, \underbrace{2, M^k, \ldots, M^k, 1, 1}}_{j \ \textit{times}}$$

*for $j \in [1, t]$;*

$(R_2)$

$$2, \underbrace{\underbrace{M^k, \ldots, M^k, 1, 1, 2}, \ldots, \underbrace{M^k, \ldots, M^k, 1, 1, 2}}_{j \ \textit{times}}, \underbrace{M^k, M^k, \ldots, M^k}_{i \ \textit{times}}$$

*for $j \in [0, t-1]$ and $i \in [1, l]$;*

($R_3$)

$$2, \underbrace{\underbrace{M^k, \ldots, M^k, 1, 1, 2}, \underbrace{M^k, \ldots, M^k, 1, 1, 2}, \ldots, \underbrace{M^k, \ldots, M^k, 1, 1, 2}}_{j \ times}$$

*for* $j \in [1, t]$;

($R_4$)

$$\underbrace{\underbrace{M^k, \ldots, M^k, 1, 1}, 2, \underbrace{M^k, \ldots, M^k, 1, 1}, 2, \ldots, 2, \underbrace{M^k, \ldots, M^k, 1, 1}}_{j \ times}$$

*for* $j \in [1, t]$.

**Proposition 1.** *Runs* ($R_1$) *and* ($R_3$) *give the same elements.*

*Proof.* The set $A$ can be written explicitly as follows:

$$A = (0 \mid 1, 1, 2, 1, \underbrace{\underbrace{4, \ldots, 4}_{k \text{ times}}, 3, \ldots, 1, \underbrace{4, \ldots, 4}_{k \text{ times}}, 3}_{l \text{ times}}, 1, 1, 2, 1, \underbrace{\underbrace{4, \ldots, 4}_{k \text{ times}}, 3, \ldots, 1, \underbrace{4, \ldots, 4}_{k \text{ times}}, 3}_{l \text{ times}},$$

$$\ldots, 1, 1, 2, 1, \underbrace{\underbrace{4, \ldots, 4}_{k \text{ times}}, 3, \ldots, 1, \underbrace{4, \ldots, 4}_{k \text{ times}}, 3}_{l \text{ times}}, 1, 1, 2, 1).$$

Since we repeat the block, it is enough to consider all possible runs starting from any block.

`Case 1.` A run starts at the first 1 in $1, 1, 2, 1$. Since we have $1 + 1 + 2 + 1 = 5 < 6$, the run must contain $1, 1, 2, 1$ and proceed further.

(i) If the run ends at 4, then it gives a sum that is congruent to $1 \pmod 4$.

(ii) If the run ends at 3, then it gives a sum that is congruent to $0 \pmod 4$.

(iii) If the run ends at $3, 1$, then it gives a sum that is congruent to $1 \pmod 4$.

(iv) If the run ends at $3, 1, 1$, then it gives a sum that is congruent to $6 \pmod 4$. In this case, the run belongs to the form ($R_1$).

(v) If the run ends at $3, 1, 1, 2$, then it gives a sum that is congruent to $0 \pmod 4$.

(vi) If the run ends at $3, 1, 1, 2, 1$, then it gives a sum that is congruent to $1 \pmod 4$.

`Case 2.` A run starts at the second 1 in $1, 1, 2, 1$. Since we have $1 + 2 + 1 = 4 < 6$, the run must contain $1, 2, 1$ and proceed further. Using similar arguments as in Case 1, we can prove that no run gives a sum that is congruent to $6 \pmod 4$.

`Case 3.` A run starts at 2 in $1, 1, 2, 1$. Since we have $2 + 1 = 3 < 6$, the run must contain $2, 1$ and proceed further. Using similar arguments as in Case 1, we can prove that if the run ends at 3 or $3, 1, 1, 2$, then it gives that the run belongs to the form $(R_2)$ and the form $(R_3)$, respectively.

`Case 4.` A run starts at $1, 4$ in $\underbrace{M^k, \ldots, M^k}_{l \text{ times}}$. Since we have $1 + 4 = 5 < 6$, the run must contain $1, 4$ and proceed further. Using similar arguments as in Case 1, we can prove that if the run ends at $3, 1, 1$, then it gives that the run belongs to the form $(R_4)$.

`Case 5.` A run starts at 4 or $3, 1, 4$ or $3, 1, 1$ in $\underbrace{M^k, \ldots, M^k}_{l \text{ times}}, 1, 1, 2, \underbrace{M^k, \ldots, M^k}_{l \text{ times}}$. Using similar arguments as in Case 1, we can prove that no run gives a sum that is congruent to $6 \pmod 4$. If we consider any run from any other block, then it is one of the above cases. Thus, we have considered all the possible cases, which completes the proof of the lemma. $\square$

**Lemma 3.** *Let $k$, $l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1, 1, 2, M^k, \ldots, M^k}, \ldots, \underbrace{1, 1, 2, M^k, \ldots, M^k}}_{t \text{ times}}, 1, 1, 2, 1).$$

*Then $(A - A)^+$ contains exactly $lt + t$ elements of the form $6 + 4m$.*

*Proof.* By Lemma 2, we know all the possible runs that can generate $(6+4m)$-type elements of $(A - A)^+$. Therefore, runs $(R_1)$ and $(R_3)$ generate the set

$$S_1 = \{6 + (4kl + 4l)j + 4(j - 1) \mid 1 \le j \le t\}.$$

Run $(R_2)$ generates the set

$$S_2 = \{6 + (4kl + 4l + 4)(j - 1) + 4ki + 4(i - 1) \mid 1 \le i \le l, 1 \le j \le t\},$$

and $(R_4)$ generates the set

$$S_3 = \{6 + (4kl + 4l)j + 4(j - 2) \mid 1 \le j \le t\}.$$

We claim that $S_1 \cap S_2 = \phi$. If this claim is not true, then there exist $j_1, j_2 \in [1, t]$ and $i_1 \in [1, l]$ such that

$$6 + (4kl + 4l)j_1 + 4(j_1 - 1) = 6 + (4kl + 4l + 4)(j_2 - 1) + 4ki_1 + 4(i_1 - 1).$$

This gives $(kl + l + 1)(j_1 - j_2 + 1) = i_1(k + 1)$. It follows that $(k + 1)l + 1$ divides $i_1(k + 1)$, but we know that $i_1(k + 1) \leq l(k + 1)$. This contradiction shows that $S_1 \cap S_2 = \phi$.

Next, we claim that $S_1 \cap S_3 = \phi$. If this claim is not true, then there exist $j_1, j_2 \in [1, t]$ such that

$$6 + (4kl + 4l)j_1 + 4(j_1 - 1) = 6 + (4kl + 4l)j_2 + 4(j_2 - 2).$$

This gives $(j_1 - j_2)(kl + l + 1) = -1$. However, we already have $(kl + l) + 1 > 1$. This contradiction shows that $S_1 \cap S_3 = \phi$.

Finally, we claim that $|S_2 \cap S_3| = t$. To see that this claim is true, it can be easily seen that for $i = l$ and $j \in [1, t]$, we have

$$6 + (4kl + 4l + 4)(j - 1) + 4kl + 4(l - 1) = 6 + (4kl + 4l)j + 4(j - 2).$$

Therefore $|S_2 \cap S_3| \geq t$. If there exist $j_1, j_2 \in [1, t]$ and $i_1 \in [1, l - 1]$ such that

$$6 + (4k + 4l + 4)(j_1 - 1) + 4ki_1 + 4(i_1 - 1) = 6 + (4kl + 4l)j_2 + 4(j_2 - 2),$$

then $(kl + l + 1)(j_2 - j_1 + 1) = i_1(k + 1) + 1$. It follows that $kl + l + 1$ divides $i_1(k + 1) + 1$, but we know that $i_1(k + 1) + 1 < l(k + 1) + 1$. This contradiction shows that $|S_2 \cap S_3| = t$. Hence,

$$|S_1 \cup S_2 \cup S_3| = t + lt + t - t = t + tl.$$

This completes the proof of the lemma. $\square$

**Theorem 6.** *Let $k, l$, and $t \in \mathbb{N}$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1, 1, 2, M^k, \ldots, M^k}, \ldots, \underbrace{1, 1, 2, M^k, \ldots, M^k}}_{t\ \ times}, 1, 1, 2, 1).$$

*Then $|A - A| = 11 + 6klt + 8lt + 8t$.*

*Proof.* It is easy to check that $1, 2 \in (A - A)^+$. Combining Lemma 1 and Lemma 3, we get

$$|A - A| = 6klt + 8lt + 8t + 11.$$

$\square$

### 3.2. Sum Set

**Lemma 4.** *Let $k, l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1, 1, 2, M^k, \ldots, M^k}, \ldots, \underbrace{1, 1, 2, M^k, \ldots, M^k}}_{t\ \ times}, 1, 1, 2, 1).$$

*Then the following hold:*

(1) $1+4i \in A$, *where* $i \in [0, 1+t(kl+l+1)]$,

(2) $2+4t(kl+l+1) \in A$,

(3) $4+4t(kl+l+1) \in A$.

*Proof.* Observe that the set of partial sums of the run

$$0, \underbrace{\underbrace{1,1,2,M^k,\ldots,M^k},\ldots,\underbrace{1,1,2,M^k,\ldots,M^k}}_{t \text{ times}}, 1,1,2,1$$

contains all elements from 1 through $5+4t(kl+l+1)$ that are congruent to 1 modulo 4 . This proves Item (1) of the lemma. Similarly, we obtain Item (2) and Item (3) of the lemma by considering the runs

$$0, \underbrace{\underbrace{1,1,2,M^k,\ldots,M^k},\ldots,\underbrace{1,1,2,M^k,\ldots,M^k}}_{t \text{ times}}, 1,1$$

and

$$0, \underbrace{\underbrace{1,1,2,M^k,\ldots,M^k},\ldots,\underbrace{1,1,2,M^k,\ldots,M^k}}_{t \text{ times}}, 1,1,2,$$

respectively. This completes the proof of the lemma. □

**Lemma 5.** *Let $k, l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1,1,2,M^k,\ldots,M^k},\ldots,\underbrace{1,1,2,M^k,\ldots,M^k}}_{t \textit{ times}}, 1,1,2,1).$$

*Then the following hold:*

(1) $1+4i \in A+A$, *where* $i \in [0, 2+2t(kl+l+1)]$,

(2) $2+4i \in A+A$, *where* $i \in [0, 2+2t(kl+l+1)]$,

(3) $3+4i \in A+A$, *where* $i \in [0, 1+2t(kl+l+1)]$.

*Proof.* By Lemma 4, we have $1+4i \in A$, for $i \in [0, 1+(kl+l+1)t]$. Therefore

$$1+4i = 0+(1+4i) \in A+A,$$

$$2+4i = 1+(1+4i) \in A+A,$$

and

$$3+4i = 2+(1+4i) \in A+A$$

for $i \in [0, 1+t(kl+l+1)]$. Also, for $j \in [1+t(kl+l+1), 2+2t(kl+l+1)]$, we have

$$1 + 4j = (1 + 4j - (4 + 4t(kl + l + 1)) + (4 + 4t(kl + l + 1)) \in A + A,$$

$$2 + 4j = (2 + 4j - (5 + 4t(kl + l + 1)) + (5 + 4t(kl + l + 1)) \in A + A,$$

and for $j \in [1 + t(kl + l + 1), 1 + 2t(kl + l + 1)]$, we have

$$3 + 4j = (3 + 4j - (2 + 4t(kl + l + 1)) + (2 + 4t(kl + l + 1)) \in A + A.$$

This completes the proof of the lemma. □

Now we consider elements divisible by 4 in $A + A$.

**Lemma 6.** *Let $k, l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1, 1, 2, M^k, \ldots, M^k}, \ldots, \underbrace{1, 1, 2, M^k, \ldots, M^k}}_{t \ \ times}, 1, 1, 2, 1).$$

*Then for $i \in [0, l]$ and $j \in [0, t-1]$,*

$$4 + (4k + 4)i + (4kl + 4l + 4)j \in A.$$

*Proof.* It is easy to see that the run

$$0, \underbrace{\underbrace{1, 1, 2, M^k, \ldots, M^k}, \ldots, \underbrace{1, 1, 2, M^k, \ldots, M^k}}_{j \ \text{times}}, 1, 1, 2, \underbrace{M^k, \ldots, M^k}_{i \ \text{times}},$$

gives

$$4 + (4k + 4)i + (4kl + 4l + 4)j \in A,$$

where $j \in [0, t-1]$ and $i \in [0, l]$. This completes the proof of the lemma. □

**Lemma 7.** *Let $k, l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1, 1, 2, M^k, \ldots, M^k}, \ldots, \underbrace{1, 1, 2, M^k, \ldots, M^k}}_{t \ \ times}, 1, 1, 2, 1).$$

*Then*

$$\{0, 4, 8\} \cup X_1 \cup X_2 \cup X_3 \cup X_4 \cup X_5 \cup X_6 \subset A + A,$$

*where*

$$X_1 = \{x_{i,j} : i \in [1, l], j \in [0, t-1]\} \ \textit{with} \ x_{i,j} = (8kl + 8l + 8)j + (4k + 4)i + 4,$$

$$X_2 = \{y_{i,j} : i \in [1, l], j \in [0, t-1]\} \ \textit{with} \ y_{i,j} = (8kl + 8l + 8)j + (4k + 4)i + 8,$$

$$X_3 = \{z_j : j \in [0, t-1]\} \textit{ with } z_j = (8kl + 8l + 8)j + (4kl + 4l) + 12,$$

$$X_4 = \{p_{i,j} : i \in [1, l], j \in [0, t-1]\} \textit{ with } p_{i,j} = (8kl+8l+8)j+(4kl+4l)+(4k+4)i+8,$$

$$X_5 = \{q_{i,j} : i \in [1, l], j \in [0, t-1]\} \textit{ with } q_{i,j} = (8kl+8l+8)j+(4kl+4l)+(4k+4)i+12,$$

*and*

$$X_6 = \{r_j : j \in [0, t-1]\} \textit{ with } r_j = (8kl + 8l + 8)j + (8kl + 8l) + 16.$$

*Also, we have* $X_i \cap X_j = \phi$ *for* $1 \leq i < j \leq 6$.

*Proof.* Using Lemma 4 and Lemma 6, we have

$$\{a_{i,j} : i \in [0, l], j \in [0, t-1]\} \cup \{a_{0,t}\} \subset A,$$

where

$$a_{i,j} = 4 + (4k + 4)i + (4kl + 4l + 4)j,$$

and

$$a_{0,t} = 4(1 + t(kl + l + 1)).$$

We have the following:

1. Since $0, 4 \in A$, $\{0, 4, 8\} \subset A + A$.
2. For $j = 0$, $x_{i,0} = 4 + (4k + 4)i = 0 + a_{i,0} \in A + A$, and for $j \geq 1$, $x_{i,j} = a_{i,j-1} + a_{l,j} \in A + A$.
3. For $j = 0$, $y_{i,0} = a_{0,0} + a_{i,0} \in A + A$, and for $j \geq 1$, $y_{i,j} = a_{i,j-1} + a_{0,j+1} \in A + A$.
4. For $j \geq 0$, $z_j = a_{0,j} + a_{0,j+1} \in A + A$.
5. For $j \geq 0$, $p_{i,j} = a_{i,j} + a_{l,j} \in A + A$.
6. For $j \geq 0$, $q_{i,j} = a_{i,j} + a_{0,j+1} \in A + A$.
7. For $j \geq 0$, $r_j = a_{0,j+1} + a_{0,j+1} \in A + A$.

Note that for $j \in [0, t-1]$,

$$x_{1,j} < y_{1,j} < x_{2,j} < \cdots < x_{l,j} < y_{l,j} < z_j < p_{1,j} < q_{1,j} < p_{2,j} < \cdots < p_{l,j} < q_{l,j} < r_j,$$

and for $t \geq 2$ and $j \in [0, t-2]$,

$$r_j < x_{1,j+1}.$$

Therefore, all these elements in $X_i$'s and $\{0, 4, 8\}$ are distinct. This completes the proof of the lemma. $\square$

**Theorem 7.** *Let $k, l$, and $t$ be positive integers. Let $A$ be*

$$(0 \mid \underbrace{\underbrace{1,1,2,M^k,\ldots,M^k},\ldots,\underbrace{1,1,2,M^k,\ldots,M^k}}_{t\ \ times},1,1,2,1).$$

*Then*

$$|A+A| \geq 11 + 6klt + 10lt + 8t.$$

*Proof.* By Lemma 4, Lemma 5, and Lemma 7, we get

$$|A+A| \geq 11 + 6klt + 10lt + 8t.$$

□

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