EFFECTIVE ESTIMATES FOR SOME FUNCTIONS DEFINED OVER PRIMES

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Abstract

In this paper we give effective estimates for some classical arithmetic functions defined over prime numbers. First we find the smallest real number x_0 so that some inequality involving Chebyshev's ϑ-function holds for every $x \geq x_0$. Then we give some new results concerning the existence of prime numbers in short intervals. Also we derive new upper and lower bounds for some functions defined over prime numbers, for instance the prime counting function $\pi(x)$, which improve current best estimates of similar shape.

1. Introduction

First, we consider Chebyshev's ϑ-function $\vartheta(x) = \sum_{p \leq x} \log p$, where p runs over all primes not exceeding x. Since there are infinitely many primes, we have $\vartheta(x) \to \infty$ as $x \to \infty$. Hadamard [37] and de la Vallée-Poussin [23] independently proved a result concerning the asymptotic behavior for $\vartheta(x)$, namely

$$\vartheta(x) \sim x \qquad (x \to \infty), \tag{1}$$

which is known as the *Prime Number Theorem*. In a later paper [24], where the existence of a zero-free region for the Riemann zeta function to the left of the line $\mathrm{Re}(s) = 1$ was proved, de la Vallée-Poussin also estimated the error term in the Prime Number Theorem by showing that

$$\vartheta(x) = x + O(xe^{-c_0\sqrt{\log x}}) \qquad (x \to \infty), \tag{2}$$

where c_0 is a positive absolute constant. The current best explicit version of this result is due to Fiori, Kadiri, and Swidinsky [34, Corollary 14]. They found that

$$|\vartheta(x) - x| \leq 121.0961 \left(\frac{\log x}{R} \right)^{3/2} \exp\left(-2\sqrt{\frac{\log x}{R}} \right) \tag{3}$$

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for every $x \geq 2$, where $R = 5.5666305$. The work of Korobov [42] and Vinogradov [71] implies the current asymptotically strongest error term in (1), namely

$$\vartheta(x) = x + O\left(x \exp\left(-c_1 \log^{3/5} x (\log\log x)^{-1/5}\right)\right) \qquad (x \to \infty), \tag{4}$$

where c_1 is a positive absolute constant. An explicit version of (4) was recently given by Johnston and Yang [40, Theorem 1.4]. Now, (2)–(4) each imply that for every positive integer k and every positive real number η_k there is real number $x_1 = x_1(k, \eta_k) > 1$ so that for every $x \geq x_1$, we have

$$|\vartheta(x) - x| < \frac{\eta_k x}{\log^k x}. \tag{5}$$

In the case where $k = 3$ and $\eta_3 = 0.024334$, Broadbent et al. [12] found that

$$|\vartheta(x) - x| < \frac{0.024334x}{\log^3 x} \qquad (x \geq e^{29}). \tag{6}$$

In our first result, we compute the smallest positive integer N so that (6) holds for every $x \geq N$.

Proposition 1. *The inequality (6) holds for every* $x \geq 1,757,126,630,797 = p_{64,707,865,143}$.

Estimates for $\vartheta(x)$ of the form (5) can be used to specify short intervals containing at least one prime number. Here, we find the following result.

Theorem 1. *Let* $a = (1.42969 \times 10^{12} - 1)^{-1}$ *and* $b = (1.59753 \times 10^{12} - 1)^{-1}$. *Further, let n be a positive integer with $1 \leq n \leq 5$. Then there is a prime number p such that*

$$x < p \leq x\left(1 + \frac{a_n}{\log^n x}\right)$$

for every $x \geq X_n$, where a_n and X_n are given as in Table 1.

n	a_n	X_n
1	$43a = 3.00\ldots \times 10^{-11}$	$952,527,672,606,693$
2	$46^n b = 1.32\ldots \times 10^{-9}$	$684,943,746,324,434$
3	$46^n b = 6.09\ldots \times 10^{-8}$	$543,684,371,469,081$
4	$46^n b = 2.802\ldots \times 10^{-6}$	$336,149,866,771,577$
5	$46^n b = 1.289\ldots \times 10^{-4}$	$246,782,656,239,427$

Table 1: Explicit values for a_n and X_n.

Remark 1. Note that the values of X_n, where $1 \leq n \leq 5$, are the smallest positive integers so that there is always a prime number in the interval $(x, x(1+a_n/\log^n x)]$. For $n \geq 6$, we are only able to find $X_n^{(1)}$ and $X_n^{(2)}$, so that analogous results are valid for all $x \in \mathbb{R}$ with $X_n^{(1)} \leq x \leq X_n^{(2)}$. To prove these results for every $x > X_n^{(2)}$ as well, we would need estimates of the form $\vartheta(x) > x - \eta_n x/\log_n x$ with $n \geq 6$.

Let $\pi(x)$ denote the number of primes not exceeding x. Chebyshev's ϑ-function and the prime counting function $\pi(x)$ are connected by the identity

$$\pi(x) = \frac{\vartheta(x)}{\log x} + \int_2^x \frac{\vartheta(t)}{t \log^2 t} \, \mathrm{d}t, \tag{7}$$

which holds for every $x \geq 2$ (see [2, Theorem 4.3]). If we combine (3) and (7), we see that

$$\pi(x) = \mathrm{li}(x) + O(xe^{-c_2\sqrt{\log x}}) \qquad (x \to \infty), \tag{8}$$

where c_2 is a positive absolute constant. Here, the *integral logarithm* $\mathrm{li}(x)$ is defined for every $x \geq 0$ as

$$\mathrm{li}(x) = \int_0^x \frac{\mathrm{d}t}{\log t} = \lim_{\varepsilon \to 0+} \left\{ \int_0^{1-\varepsilon} \frac{\mathrm{d}t}{\log t} + \int_{1+\varepsilon}^x \frac{\mathrm{d}t}{\log t} \right\}$$

and plays an important role in this paper. The current best explicit version of (8) is due to Johnston and Yang [40, Corollary 1.3]. Again, the work of Korobov [42] and Vinogradov [71] implies the current asymptotically strongest error term for the difference $\pi(x) - \mathrm{li}(x)$, namely

$$\pi(x) = \mathrm{li}(x) + O\left(x \exp\left(-c_3(\log x)^{3/5}(\log\log x)^{-1/5}\right)\right) \qquad (x \to \infty), \tag{9}$$

where c_3 is a positive absolute constant. Ford [36, p. 2] has found that the constant c_3 in (9) can be chosen to be equal to 0.2098. Johnston and Yang [40, Theorem 1.4] used explicit zero-free regions and zero-density estimates for the Riemann zeta-function to show that the inequality

$$|\pi(x) - \mathrm{li}(x)| \leq 0.028x(\log x)^{0.801} \exp\left(-0.1853(\log x)^{3/5}(\log\log x)^{-1/5}\right) \tag{10}$$

holds for every $x \geq 71$. Panaitopol [53, p. 55] gave another completely different asymptotic formula for the prime counting function by showing that for every positive integer m, one has

$$\pi(x) = \frac{x}{\log x - 1 - \frac{k_1}{\log x} - \frac{k_2}{\log^2 x} - \ldots - \frac{k_m}{\log^m x}} + O\left(\frac{x}{\log^{m+2} x}\right) \qquad (x \to \infty), \tag{11}$$

where the positive integers $k_1, \ldots, k_m$ are defined by the recurrence formula

$$k_m + 1!k_{m-1} + 2!k_{m-2} + \ldots + (m-1)!k_1 = m \cdot m!.$$

For instance, we have $k_1 = 1$, $k_2 = 3$, $k_3 = 13$, $k_4 = 71$, $k_5 = 461$, and $k_6 = 3441$. The computation of the prime counting function $\pi(x)$ for large values of x is a difficult problem (the latest record is due to Baugh and Walisch and was $\pi(10^{28}) = 157,589,269,275,973,410,412,739,598$). Also the asymptotic formula (8) (or (11)) is not very meaningful with regard to the computation of $\pi(x)$ for some fixed x. Hence we are interested in finding new effective estimates for the prime counting function $\pi(x)$ which correspond to the first terms of (11). For instance, those estimates for the prime counting function are used to get effective estimates for $1/\pi(x)$ (see [10]) or the nth prime number (see [7]). In this paper, we use Proposition 1 to establish the following upper bound for $\pi(x)$ which corresponds to the first terms of the asymptotic formula (11).

Theorem 2. *Let $a_5 = 461.364417856444$ and $a_6 = 4331.1$. Then for every $x \geq 48$, we have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.024334}{\log^2 x} - \frac{12.975666}{\log^3 x} - \frac{71.048668}{\log^4 x} - \frac{a_5}{\log^5 x} - \frac{a_6}{\log^6 x}}. \tag{12}$$

For all sufficiently large values of x, Theorem 2 is a consequence of (10). On the other hand, we get the following lower bound for the $\pi(x)$ which corresponds to the first terms of (11).

Theorem 3. *Let $b_5 = 460.634397856444$ and $b_6 = 3444.031844143556$. Then for every $x \geq 1,751,189,194,177 = p_{64,497,259,289}$, we have*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.975666}{\log^2 x} - \frac{13.024334}{\log^3 x} - \frac{70.951332}{\log^4 x} - \frac{b_5}{\log^5 x} - \frac{b_6}{\log^6 x}}. \tag{13}$$

Again, for all sufficiently large values of x, Theorem 3 follows directly from (10). The asymptotic expansion (11) implies that the slightly sharper inequality

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x}} \tag{14}$$

holds for all sufficiently large values of x. In [6, Theorem 1], the present author was able to prove that the inequality (14) holds for every x such that $65\,405\,887 \leq x \leq 2.7358 \cdot 10^{40}$ and every $x \geq 4.8447 \cdot 10^{19377}$. Under the assumption that the Riemann hypothesis is true, the present author[6, Proposition 2] showed that the inequality (14) holds for every $x \geq 65,405,887$. In the following theorem we finally see that the inequality (14) holds for every $x \geq 65,405,887$ even without the assumption that the Riemann hypothesis is true.

Theorem 4. *The inequality (14) holds unconditionally for every $x \geq 65,405,887$.*

Our next goal is to establish new explicit estimates for the functions

$$\sum_{p \leq x} \frac{1}{p} \quad \text{and} \quad \sum_{p \leq x} \frac{\log p}{p},$$

where p runs over primes not exceeding x, respectively. Euler [32] proved that the sum of the reciprocals of all prime numbers diverges. Mertens [49, p. 52] found that $\log\log x$ is the right order of magnitude for this sum by showing

$$\sum_{p\leq x}\frac{1}{p}=\log\log x+B+O\left(\frac{1}{\log x}\right). \tag{15}$$

Here B denotes *the Mertens' constant* and is defined by

$$B=\gamma+\sum_{p}\left(\log\left(1-\frac{1}{p}\right)+\frac{1}{p}\right)=0.26149\ldots, \tag{16}$$

where $\gamma=0.577215\ldots$ denotes the Euler-Mascheroni constant. In Section 6, we apply Proposition 1 to some identity obtained by Rosser and Schoenfeld [63] and derive the following result which improves all other results of this form.

Theorem 5. *For every $x\geq 1,757,126,630,797$, we have*

$$\left|\sum_{p\leq x}\frac{1}{p}-\log\log x-B\right|\leq\frac{0.024334}{3\log^3 x}\left(1+\frac{15}{4\log x}\right). \tag{17}$$

In 1874, Mertens [49] showed that

$$\sum_{p\leq x}\frac{\log p}{p}=\log x+O(1). \tag{18}$$

Landau [45, §55] improved (18) by finding

$$\sum_{p\leq x}\frac{\log p}{p}=\log x+E+O(\exp(-\sqrt[14]{\log x})),$$

where E is a constant defined by

$$E=-\gamma-\sum_{p}\frac{\log p}{p(p-1)}=-1.3325\ldots. \tag{19}$$

Similar to Theorem 5, we establish the following explicit estimates for $\sum_{p\leq x}\log(p)/p$ which improve [5, Proposition 8].

Theorem 6. *For every $x\geq 1,757,126,630,797$, we have*

$$\left|\sum_{p\leq x}\frac{\log p}{p}-\log x-E\right|\leq\frac{0.024334}{2\log^2 x}\left(1+\frac{2}{\log x}\right).$$

Remark 2. Note that the positive integer $N_0=1,757,126,630,797$ in Theorem 6 might not be the smallest positive integer N so that the inequality given in Theorem 6 holds for every $x\geq N$.

2. Proof of Proposition 1

In the following proof of Proposition 1, we first utilize an identity investigated by Rosser and Schoenfeld [63] to express Chebyshev's ϑ-function in terms of the difference $\pi(x) - \mathrm{li}(x)$. Then we apply Walisch's *primecount* C++ code [74] to find a lower bound for $\pi(x) - \mathrm{li}(x)$ in a certain restricted interval.

Proof of Proposition 1. By (6) and [12, Corollary 11.1], it suffices to check that the inequality

$$\vartheta(x) > x - \frac{0.024334x}{\log^3 x} \tag{20}$$

holds for every x satisfying $1,757,126,630,797 \leq x \leq e^{29}$. Using [63, (2.26)] with $f(x) = \log x$, we get

$$\vartheta(x) = x - 2 + \mathrm{li}(2) \log 2 + (\pi(x) - \mathrm{li}(x)) \log x - \int_2^x \frac{\pi(t) - \mathrm{li}(t)}{t} \, \mathrm{d}t \tag{21}$$

for every $x \geq 2$. Now we can use [54, Corollary 1] to see that

$$-2 + \mathrm{li}(2) \log 2 - \int_2^x \frac{\pi(t) - \mathrm{li}(t)}{t} \, \mathrm{d}t \geq -2 + \mathrm{li}(2) \log 2 - \int_2^9 \frac{\pi(t) - \mathrm{li}(t)}{t} \, \mathrm{d}t \geq 0.129 \tag{22}$$

for every x with $9 \leq x \leq e^{29}$. Applying (23) to (21), we get

$$\vartheta(x) > x + (\pi(x) - \mathrm{li}(x)) \log x \tag{23}$$

for every x so that $9 \leq x \leq e^{29}$. Now we use Walisch's *primecount* C++ code [74] to get

$$\pi(x) - \mathrm{li}(x) \geq -\frac{0.024334x}{\log^4 x} \tag{24}$$

for every x with $1,760,505,892,241 \leq x \leq 2,342,911,050,819$ and every x with $2,346,094,807,193 \leq x \leq 4 \times 10^{12}$. If we combine (24) with (23), we get (20) for every x satisfying $1,760,505,892,241 \leq x \leq 2,342,911,050,819$ and every x with $2,346,094,807,193 \leq x \leq e^{29} \leq 4 \times 10^{12}$. In order to verify the required inequality (20) in the case where x satisfies $1,757,126,630,797 \leq x < 1,760,505,892,241$, we can check with a computer that $\vartheta(p_n) > g(p_{n+1})$ for every integer n such that $\pi(1,757,126,630,797) \leq n \leq \pi(1,760,505,892,241)$. Finally, a direct computer check shows that the inequality (20) also holds for every x such that $2,342,911,050,819 \leq x \leq 2,346,094,807,193$. □

The present author [5, Theorem 1, Proposition 1, and Equations (4.4) and (4.5)] utilized [30, Table 1 and Corollary 4.5] to show that

$$|\vartheta(x) - x| < \frac{0.043x}{\log^3 x} \qquad (x \geq e^{40}), \tag{25}$$

$$|\vartheta(x) - x| < \frac{0.15x}{\log^3 x} \qquad (e^{35} \leq x < e^{5000}), \tag{26}$$

$$|\vartheta(x) - x| < \frac{99.07x}{\log^4 x} \qquad (x \geq e^{25}), \tag{27}$$

$$|\vartheta(x) - x| < \frac{100x}{\log^4 x} \qquad (x \geq 70,111). \tag{28}$$

Broadbent et al. [12, p. 2299] pointed out that the main theorem of [59] is incorrect and thus bounds claimed in [30] are likely affected, in particular [30, Table 1] for bounds for $\psi(x)$, and consequently the inequalities (25)–(28). Except for the corresponding line for the value $b = 2500$, all other explicit values in [30, Table 1] were confirmed and even improved by Broadbent et al. [12, Table 8] while the corresponding line for the value $b = 2500$ was recently confirmed and even improved by Fiori, Kadiri, and Swidinsky [33, Table 5]. Hence, we can recover [30, Table 1].

Proposition 2. *The explicit values for ε given in [30, Table 1] are correct.*

To show that the inequalities (25)–(28) still hold, it suffices to note that [30, Proposition 4.4] combined with [12, Proposition 4] yield the correctness of [30, Corollary 4.5]. Hence, we get

Proposition 3. *The inequalities (25)–(28) for Chebyshev's ϑ-function are correct.*

Remark 3. Note that Proposition 1 already provides the correctness of the inequalities (25) and (26).

Remark 4. To find other explicit estimates for $\vartheta(x)$ in the restricted interval $[2, 10^{20}]$, one can also apply the method used by Dusart in [31]. Let

$$\pi_0(x) = \lim_{\varepsilon \to 0} \frac{\pi(x-\varepsilon) + \pi(x+\varepsilon)}{2} = \begin{cases} \pi(x) - 1/2, & \text{if } x \text{ is prime,} \\ \pi(x), & \text{otherwise.} \end{cases}$$

Riemann [60] published the formula

$$\pi_0(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} f(x^{1/n}), \tag{29}$$

where $\mu(n)$ is the Möbius function, and $f(x)$ is the Riemann prime counting function

$$f(x) = \mathrm{li}(x) - \sum_{\rho} \mathrm{li}(x^{\rho}) + \int_x^{\infty} \frac{\mathrm{d}t}{t(t^2-1)\log t} - \log 2.$$

Here the sum means $\lim_{T\to\infty}\sum_{|\rho|\le T}\operatorname{li}(x^\rho)$, and the ρ's are the nontrivial zeros of the Riemann zeta function. A first proof of (29) was given by von Mangoldt [73] in 1895. Now let

$$R(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \operatorname{li}(x^{1/n}) = 1 + \sum_{k=1}^{\infty} \frac{\log^k x}{k!k\zeta(k+1)}. \tag{30}$$

The latter series for it is known as Gram series. Since $\log x < x$ for every real $x > 0$, this series converges for all positive x by comparison with the series for e^x. In [61], Riesel and Göhl showed that the function

$$g(x) = R(x) - \frac{1}{\log x} + \frac{1}{\pi} \arctan \frac{\pi}{\log x}$$

is a quite good approximation to $\pi_0(x)$. The difference between $g(x)$ and $\pi_0(x)$ heuristically oscillates with an amplitude of about $\sqrt{x}/\log x$. So we define

$$\Delta(x) = \left(\pi_0(x) - R(x) + \frac{1}{\log x} - \frac{1}{\pi} \arctan \frac{\pi}{\log x}\right) \frac{\log x}{\sqrt{x}}, \tag{31}$$

the function which represents the fluctuations of the distribution of primes. We can use (30) and (31) to get

$$\pi(x) - \operatorname{li}(x) \le \frac{1}{2} + f_2(x) + \frac{\sqrt{x}}{\log x} \times \Delta(x) - \frac{1}{\log x} + \frac{1}{\pi} \arctan \frac{\pi}{\log x}, \tag{32}$$

where

$$f_k(x) = \sum_{n=k}^{\infty} \frac{\mu(n)}{n} \operatorname{li}(x^{1/n}).$$

Since $\mu(4) = 0$ and $f_5(x)$ is strictly decreasing on $(1, \infty)$, the inequality (32) implies that

$$\pi(x) - \operatorname{li}(x) \le -\frac{\operatorname{li}(\sqrt{x})}{2} - \frac{\operatorname{li}(x^{1/3})}{3} + \frac{\sqrt{x}}{\log x} \times \Delta(x) \tag{33}$$

for every $x \ge 2,000$. Similarly, we see that

$$\pi(x) - \operatorname{li}(x) \ge \sum_{n=2}^{5} \frac{\mu(n)}{n} \operatorname{li}(x^{1/n}) + \frac{\sqrt{x}}{\log x} \times \Delta(x) \tag{34}$$

for every $x \ge 10,326$. Applying (33) and (34) to (21), we get

$$\vartheta(x) > x + (\Delta(x) - 1)\sqrt{x} - \max_{2000 \le t \le x} \Delta(t) \times \operatorname{li}(\sqrt{x}) - \sqrt[3]{x} - \frac{\operatorname{li}(\sqrt[5]{x}) \log x}{5} + c_1$$

for every $x \ge 10,326$, where c_1 is a constant. Analogously, we see that the inequality

$$\vartheta(x) < x + (\Delta(x) - 1)\sqrt{x} - \min_{10,236 \le t \le x} \Delta(t) \times \operatorname{li}(\sqrt{x}) - \sqrt[3]{x} + \frac{\operatorname{li}(\sqrt[5]{x}) \log x}{5} - \sqrt[5]{x} + c_2$$

holds for every $x \geq 10,326$, where c_2 is a constant. Now one can use the extensive table of the minimum and maximum values of $\Delta(x)$ in [44] to obtain explicit estimates for $\vartheta(x)$ in the restricted interval $[2, 10^{20}]$.

Remark 5. Under the assumption that the Riemann hypothesis is true, von Koch [72] deduced the asymptotic formula $\vartheta(x) = x + O(\sqrt{x}\log^2 x)$. An explicit version was given by Schoenfeld [66, Theorem 10]. Under the assumption that the Riemann hypothesis is true, Schoenfeld has found that

$$|\vartheta(x) - x| < \frac{\sqrt{x}}{8\pi}\log^2 x \tag{35}$$

for every $x \geq 599$. Recently, Schoenfeld's result was slightly improved by Dusart [31, Proposition 2.5]. In 2016, Büthe [14, Theorem 2] investigated a method to show that the inequality (35) holds unconditionally for every x such that $599 \leq x \leq 1.4 \times 10^{25}$. Büthe's result was improved by Platt and Trudgian [55, Corollary 1]. They proved that the inequality (35) holds unconditionally for every x satisfying $599 \leq x \leq 2.169 \times 10^{25}$. Recently, Johnston [39, Corollary 3.3] extended the last result by showing that the inequality (35) holds unconditionally for every x with $599 \leq x \leq 1.101 \times 10^{26}$.

3. Proof of Theorem 1

Bertrand's postulate states that for each positive integer n there is a prime number p with $n < p \leq 2n$. It was proved, for instance, by Chebyshev [18]. In the following, we note some improvements of Bertrand's postulate. The first result is due to Schoenfeld [66, Theorem 12]. He discovered that for every $x \geq 2,010,759.9$ there is a prime number p with $x < p < x(1 + 1/16,597)$. Ramaré and Saouter [58, Theorem 3] proved that for every $x \geq 10,726,905,041$ there is a prime number p so that $x < p \leq x(1 + 1/28,313,999)$. Further, they gave a table of sharper results which hold for large x, see [58, Table 1]. Kadiri and Lumley [41, Table 2] obtained a series of improvements. For instance, they showed that for every $x \geq 4 \times 10^{18}$ there is a prime number p such that $x < p < x(1+1/36,082,898)$. Recently, Cully-Hugill and Lee [22, Theorem 1] improved the results of Kadiri and Lumley. In particular, they found that for every $x \geq 4 \times 10^{18}$ there is a prime number p so that $x < p \leq x(1+1/1,429,689,999,999)$. Dusart [28, Théorème 1] proved that for every $x \geq 3,275$ there exists a prime number p such that $x < p \leq x(1 + 1/(2\log^2 x))$ and then reduced the interval himself [29, Proposition 6.8] by showing that for every $x \geq 396,738$ there is a prime number p satisfying $x < p \leq x(1 + 1/(25\log^2 x))$. Trudgian [70, Corollary 2] proved that for every $x \geq 2,898,242$ there exists a prime number p with

$$x < p \leq x\left(1 + \frac{1}{111\log^2 x}\right). \tag{36}$$

In [3, Theorem 1.26], it is shown that for every $x \geq 58,833$ there is a prime number p such that

$$x < p \leq x\left(1 + \frac{1.274}{\log^3 x}\right).$$

This was improved in [4, Theorem 1.5] by showing that for every $x \geq 58,837$ there is a prime number p such that $x < p \leq x(1 + 1.1817/\log^3 x)$. Dusart [30, p. 243] used (the recovered) Table 1 of [30] (cf. Proposition 2) to show the inequality

$$|\vartheta(x) - x| < \frac{0.499x}{\log^3 x} \qquad (x \geq 4 \times 10^{18}). \tag{37}$$

Alternatively, the inequality (37) follows directly from Proposition 1. Then, he [30, Proposition 5.4] utilized the inequality (37) to see that for every $x \geq 89,693$ there exists a prime number p such that

$$x < p \leq x\left(1 + \frac{1}{\log^3 x}\right) \tag{38}$$

and concluded from this that for every $x \geq 468,991,632$ there exists a prime number p such that

$$x < p \leq x\left(1 + \frac{1}{5,000\log^2 x}\right) \tag{39}$$

which improves Trudgian's result (36). In [5, Theorem 4], the present author combined (39) and the (recovered) inequality (25) (cf. Proposition 3) to obtain that for every $x \geq 6,034,256$ there exists a prime number p such that

$$x < p \leq x\left(1 + \frac{0.087}{\log^3 x}\right). \tag{40}$$

Further, the present author [5, Theorem 4] used the (recovered) inequality (27) (cf. Proposition 3) and (39) to find found that for every $x > 1$ there is a prime number p with

$$x < p \leq x\left(1 + \frac{198.2}{\log^4 x}\right). \tag{41}$$

Now we give a proof of Theorem 1 where we give improvements of (38)–(41) by decreasing the coefficient of the term $1/\log^n x$ and on the other hand by increasing the exponent of the $\log x$ term.

Proof of Theorem 1. In order to prove that there is a prime number p with $x < p \leq x(1 + a_1/\log x)$ for every $x \geq X_1 = 952,527,672,606,693$, we first consider the case where $x \geq \exp(4,000)$. Here, we can use [12, Table 15] to get that

$$\vartheta\left(x\left(1 + \frac{a_1}{\log x}\right)\right) - \vartheta(x) > \frac{x}{\log x}\left(a_1 - 2\varepsilon - \frac{a_1\varepsilon}{\log x}\right) \geq 0,$$

where $\varepsilon = 5.741 \times 10^{-13}$, which implies that for every $x \geq \exp(4,000)$ there is a prime number p satisfying $x < p \leq x(1 + a_1/\log x)$. For every x with $4 \times 10^{18} \leq x < \exp(4,000)$, the claim follows directly from [22, Theorem 1]. So it suffices to consider the case where $952,527,672,606,693 \leq x < 4 \times 10^{18}$. Let n be an integer so that $1,721,649,982,233,847 \leq p_n \leq \pi(4 \times 10^{18})$ and let x be a real number satisfying $p_n \leq x < p_{n+1}$. Then, we can utilize [51, Table 8] to see that

$$p_{n+1} - p_n \leq 1,476 < \frac{a_1 p_n}{\log p_n} \leq \frac{a_1 x}{\log x}.$$

This implies that for every x with $1,721,649,982,233,847 \leq x < 4 \times 10^{18}$ there is a prime number in the interval $(x, x(1+a_1/\log x)]$. Similar, we can see that for every x satisfying $1,041,648,882,338,903 \leq x < 1,721,649,982,233,847$ there is always a prime in the interval $(x, x(1 + a_1/\log x)]$. Next, we can use Walisch's *primesieve* program [75] to obtain that

$$p_{n+1} - p_n \leq 860 < \frac{a_1 p_n}{\log p_n} \leq \frac{a_1 x}{\log x}$$

for every integer n satisfying $9.88 \times 10^{14} \leq p_n < 1.042 \times 10^{15}$ and every x with $p_n \leq x < p_{n+1}$. So there exists a prime number p with $x < p \leq x(1 + a_1/\log x)$ for every x so that $9.88 \times 10^{14} \leq x \leq 1,041,648,882,338,903$. If n is an integer with $9.53 \times 10^{14} \leq p_n < 9.88 \times 10^{14}$ and x satisfies $p_n \leq x < p_{n+1}$, we can use Walisch's *primesieve* program [75] to see that $p_{n+1} - p_n \leq 802 < a_1 p_n/\log p_n \leq a_1 x/\log x$. This provides that for every x with $9.53 \times 10^{14} \leq x < 9.88 \times 10^{14}$ there is always a prime number in the interval $(x, x(1 + a_1/\log x)]$. For every integer n satisfying $952,527,672,607,523 \leq p_n < 9.53 \times 10^{14}$ and every x with $p_n \leq x < p_{n+1}$, we apply Walisch's *primesieve* program [75] to obtain that $p_{n+1} - p_n \leq 708 < a_1 p_n/\log p_n \leq a_1 x/\log x$ and it turns out that for every x with $952,527,672,607,523 \leq x < 9.53 \times 10^{14}$ there is a prime number in the interval $(x, x(1 + a_1/\log x)]$. Finally, it suffices to consider the case where x belongs to the interval $[a, b)$ where $a = 952,527,672,606,693$ and $b = 952,527,672,607,523$. In this situation, we have $\pi(x(1 + a_1/\log x)) - \pi(x) \geq 1$ as desired. The proof of the remaining assertions is similar to the above proof and we leave the details to the reader. □

Remark 6. Beginning with Hoheisel [38], many authors have found shorter intervals of the form $[x - x^{\delta}, x]$ that must contain a prime number for all sufficiently large values of x. The most recent result is due to Baker, Harman, and Pintz [8]. They found the value $\delta = 0.525$. Under the assumption that the Riemann hypothesis is true, much better results are known. For more details, see, for instance, Ramaré and Saouter [58], Dudek [26], Dudek, Grenié, and Molteni [27], Carneiro, Milinovich, and Soundararajan [16], Cully-Hugill and Dudek [19], Cully-Hugill and Johnston [20], and Cully-Hugill and Dudek [21].

4. Proof of Theorem 2

First, we note some well known estimates for the prime counting function $\pi(x)$. A classic method of finding explicit estimates for $\pi(x)$ is the following. Let k be a positive integer and η_k a positive real number. By (5), there is a real number $x_1 = x_1(k, \eta_k) > 1$ so that

$$|\vartheta(x) - x| < \frac{\eta_k x}{\log^k x}$$

for every $x \geq x_1$. In order to prove Theorem 2, we define the auxiliary function

$$J_{k;\eta_k;x_1}(x) = \pi(x_1) - \frac{\vartheta(x_1)}{\log x_1} + \frac{x}{\log x} + \frac{\eta_k x}{\log^{k+1} x} + \int_{x_1}^{x} \left(\frac{1}{\log^2 t} + \frac{\eta_k}{\log^{k+2} t} \, \mathrm{d}t \right) \quad (42)$$

and note the following both inequalities involving the prime counting function $\pi(x)$.

Lemma 1. *For every $x \geq x_1$, we have $J_{k;-\eta_k;x_1}(x) \leq \pi(x) \leq J_{k;\eta_k;x_1}(x)$.*

Proof. The claim follows directly form (7) and (5). □

One of the first estimates for $\pi(x)$ is due to Gauss. In 1793, he computed that

$$\pi(x) \leq \mathrm{li}(x) \quad (43)$$

holds for every x with $2 \leq x \leq 3{,}000{,}000$ and conjectured that the inequality (43) holds for every $x \geq 2$. This conjecture was disproven by Littlewood [48]. More precisely, he proved that the function $\pi(x) - \mathrm{li}(x)$ changes the sign infinitely many times. Unfortunetely, Littlewood's proof is nonconstructive and there is still no example of x such that $\pi(x) > \mathrm{li}(x)$. Skewes [67] proved the existence of a number x_0 with $x_0 < \exp(\exp(\exp(\exp(7.705))))$ such that $\pi(x_0) > \mathrm{li}(x_0)$. Lehman [47] improved this last upper bound considerably by showing that exists a number x_0 with $x_0 < 1.65 \times 10^{1165}$ such that $\pi(x_0) > \mathrm{li}(x_0)$. After some further improvements (see, for instance, te Riele [69], Bays and Hudson [9], Chao and Plymen [17], Saouter and Demichel [65], Stoll and Demichel [68]), the current best upper bound was found by Saouter, Trudgian, and Demichel [64]. They proved that there exists a number x_0 with $x_0 < \exp(727.951335621)$ such that $\pi(x_0) > \mathrm{li}(x_0)$. All these upper bounds have been proved by using computer calculations of zeros of the Riemann zeta function. The first lower bound for a number x_0 with $\pi(x_0) > \mathrm{li}(x_0)$ was given by the calculation of Gauss, namely $x_0 > 3{,}000{,}000$. This lower bound was improved in a series of papers. For details, see Rosser and Schoenfeld [63], Brent [11], Kotnik [43], Platt and Trudgian [54], and Stoll and Demichel [68]. For our further inverstigation, we use the following improvement.

Lemma 2 (Büthe [15])**.** *For every x with $2 \leq x \leq 10^{19}$, we have $\pi(x) \leq \mathrm{li}(x)$.*

Remark 7. Recently. Dusart [31, Lemma 2.2] showed that $\pi(x) \leq \mathrm{li}(x)$ for every x with $2 \leq x \leq 10^{20}$.

Now we use Proposition 1 and the Lemmata 1 and 2 to give a proof of Theorem 2.

Proof of Theorem 2. First, we combine Lemma 1 with Proposition 1 to see that

$$J_{3;-0.024334;x_1}(x) \leq \pi(x) \leq J_{3;0.024334;x_1}(x) \tag{44}$$

for every $x \geq x_1$, where $x_1 \geq 1,757,126,630,797$. Now, let $x_2 = 10^{18}$ and let $f(x)$ be given by the right-hand side of (12). We consider the function $g(x) = f(x) - J_{3,0.024334,x_2}(x)$. By [25], we have $\vartheta(x_2) \geq 999,999,999,144,115,634$. Further, $\pi(x_2) = 24,739,954,287,740,860$ and so we compute $g(x_2) \geq 2 \times 10^8$. Since the derivative of g is positive for every $x \geq x_2$, we get $f(x) - J_{3,0.024334,x_2}(x) > 0$ for every $x \geq x_1$, and we conclude from (44) that the inequality (12) holds for every $x \geq x_1$. Comparing $f(x)$ with the integral logarithm $\mathrm{li}(x)$, we see that $f(x) > \mathrm{li}(x)$ for every $x \geq 121,141,948$. Now we can utilize Lemma 2 to see that the desired inequality also holds for every x such that $121,141,948 \leq x < 10^{18}$. A computer check for smaller values of x completes the proof. □

Under the assumption that the Riemann hypothesis is true, von Koch [72] deduced that $\pi(x) = \mathrm{li}(x) + O(\sqrt{x}\log x)$ as $x \to \infty$. Actually, one can show that the asymptotic formula $\pi(x) = \mathrm{li}(x) + O(\sqrt{x}\log x)$ as $x \to \infty$ is even a sufficient criterion for the truth of the Riemann hypothesis. An explicit version of von Koch's result is due to Schoenfeld [66, Corollary 1]. Under the assumption that the Riemann hypothesis is true, Schoenfeld found that the inequality

$$|\pi(x) - \mathrm{li}(x)| < \frac{1}{8\pi}\sqrt{x}\log x \tag{45}$$

holds for every $x \geq 2,657$. In 2014, Büthe [14, p. 2,495] proved that the inequality (45) holds unconditionally for every x such that $2,657 \leq x \leq 1.4 \times 10^{25}$. Platt and Trudgian [55, Corollary 1] improved Büthe's result by showing that the inequality (45) holds unconditionally for every x satisfying $2,657 \leq x \leq 2.169 \times 10^{25}$. Johnston [39, Corollary 3.3] extended the last result by showing the following

Lemma 3 (Johnston)**.** *The inequality* (45) *holds unconditionally for every x satisfying* $2,657 \leq x \leq 1.101 \times 10^{26}$.

Now we can use Theorem 2 and the Lemmata 2 and (3) to find the following weaker but more compact upper bounds for the prime counting function $\pi(x)$ of the form

$$\pi(x) < \frac{x}{\log x - a_0 - \frac{a_1}{\log x} - \cdots - \frac{a_m}{\log^m x}} \qquad (x \geq x_0),$$

where m is a integer with $0 \leq m \leq 5$ and $a_0, \ldots, a_m$ are suitable positive real numbers.

Corollary 1. *We have*

$$\pi(x) < \frac{x}{\log x - a_0 - \frac{a_1}{\log x} - \frac{a_2}{\log^2 x}}$$

for every $x \geq x_0$, where a_0, a_1, a_2, and x_0 are given as in Table 2.

a_0	1.0343	1	1
a_1	0	1.109	1
a_2	0	0	3.48
x_0	$106,640,139,304,611$	$81,250,795,096,339$	$145,413,088,724,077$

Table 2: Explicit values for a_0, a_1, a_2, and x_0.

Proof. Theorem 2 implies that the inequality

$$\pi(x) < \frac{x}{\log x - 1.0343} \tag{46}$$

holds for every $x \geq 108,943,258,198,427$. If we compare the right-hand side of (46) with $\mathrm{li}(x)$, we can use Lemma 2 to see that the required inequality (46) holds for every x with $106,910,668,441,596 \leq x \leq 108,943,258,198,427$. Finally, we use Walisch's *primecount* program [74] to obtain that the inequality (46) is also valid for every x satisfying $106,640,139,304,611 \leq x \leq 106,910,668,441,596$. The proof of each of the next three inequalities is similar to the proof of (46) and we leave the details to the reader. Next, we show that the inequality

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.024334}{\log^2 x} - \frac{12.975666}{\log^3 x} - \frac{79.962}{\log^4 x}} \tag{47}$$

holds for every $x \geq 22$. First, we can use Theorem 2 to obtain the inequality (47) for every $x \geq 1.101 \times 10^{26}$. Let $f(x)$ denote the right-hand side of (46). We get that $f(x) \geq \mathrm{li}(x) + \sqrt{x}\log(x)/(8\pi)$ for every x with $22,066,689,219,741,110 \leq x \leq 1.101 \times 10^{26}$. Now we can apply Lemma 3 to see that the required inequality (47) also holds for every x satisfying $22,066,689,219,741,110 \leq x \leq 1.101 \times 10^{26}$. A comparison with $\mathrm{li}(x)$ shows that $f(x) > \mathrm{li}(x)$ for every $x \geq 259,576,712,645$ and Lemma 2 yields the desired inequality (47) for every x with $259,576,712,645 \leq x \leq 22,066,689,219,741,110$. Finally, it suffices to apply Walisch's *primecount* program [74] to see that the inequality (47) also holds for every x satisfying $22 \leq x \leq 259,576,712,645$. Again, the proof of the remaining inequality is similar to the proof of (47) and we leave the details to the reader. □

Remark 8. In the appendix at the end of this paper, we give lots of other weaker upper bounds in the case where $m \in \{0, 1, 2\}$.

Corollary 2. *We have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3.024334}{\log^2 x} - \frac{a_3}{\log^3 x} - \frac{a_4}{\log^4 x} - \frac{a_5}{\log^5 x}}$$

for every $x \geq x_0$, where a_3, a_4, a_5, and x_0 are given as in Table 3.

a_3	14.893	12.975666	12.975666
a_4	0	79.962	71.048668
a_5	0	0	533.594
x_0	$142,464,507,937,911$	22	32

Table 3: Explicit values for a_3, a_4, a_5, and x_0.

Proof. Since the proof is similar to the proof of Corollary 1, we leave the details to the reader. □

Using an estimate for Chebyshev's ϑ-function found by Broadbent et al. [12, Table 15], we get the following upper bound for $\pi(x)$ which improves the inequality (12) for all sufficiently large values of x.

Proposition 4. *For every $x \geq 29.53$, we have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{70.935}{\log^3 x}}.$$

Proof. The proof is similar to the proof of Theorem 2 and we leave the details to the reader. We denote the right-hand side of (4) by $f(x)$ and let $x_1 = 10^{18}$. We combine Lemma 1 with [12, Table 15] to see that $\pi(x) \leq J_{4,57.184,x_1}(x)$ for every $x \geq x_1$. So it suffices to compare $f(x)$ with $J_{4,57.184,x_1}(x)$ to get that $f(x) > \pi(x)$ for every $x \geq 10^{18}$. Since $f(x) > \mathrm{li}(x)$ for every x such that $1,098 \leq x < 10^{18}$, we can apply Lemma 2 to obtain that (4) also holds for every x such that $1,098 \leq x < 10^{18}$. A direct computation for smaller values of x completes the proof. □

Integration by parts in (8) implies that for every positive integer m, one has

$$\pi(x) = \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \ldots + \frac{(m-1)!x}{\log^m x} + O\left(\frac{x}{\log^{m+1} x}\right) \tag{48}$$

as $x \to \infty$. In this direction, we get the following upper bound for $\pi(x)$.

Proposition 5. *For every $x > 1$, we have*

$$\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6.024334x}{\log^4 x} + \frac{24.024334x}{\log^5 x} + \frac{120.12167x}{\log^6 x} + \frac{720.73002x}{\log^7 x} + \frac{6098x}{\log^8 x}.$$

Proof. We set $x_1 = 10^{18}$. Further, let $f(x)$ be the right-hand side of the required inequality. We have $f(x) > J_{3,0.024334,x_1}(x)$ for every $x \geq x_1$. So, we can use (44) to get $f(x) > \pi(x)$ for every $x \geq x_1$. Since $f(x) > \mathrm{li}(x)$ for every $x \geq 204,182,829$, we can apply Lemma 2 to obtain $f(x) > \pi(x)$ for every x such that $204,182,829 \leq x \leq x_1$. A direct computation for smaller values of x completes the proof. □

Proposition 5 yields the following weaker but more compact upper bounds for the prime counting function $\pi(x)$.

Corollary 3. *For every $x \geq x_0$, we have*

$$\pi(x) < \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{(2+\varepsilon)x}{\log^3 x},$$

where ε and x_0 are given as in Table 4.

ε	0.21	0.215	0.22
x_0	$160,930,932,942,272$	$83,016,503,500,865$	$43,999,690,220,699$
ε	0.225	0.23	0.24
x_0	$23,824,649,646,672$	$13,279,102,022,111$	$4,511,700,549,332$
ε	0.25	0.26	0.265[1]
x_0	$1,615,202,653,795$	$643,809,266,445$	$406,742,886,708$
ε	0.27	0.28	0.29
x_0	$265,248,130,170$	$117,997,473,286$	$57,720,805,589$

Table 4: Explicit values for ε and x_0.

Proof. Let $x_0 = 160,930,932,942,272$ and $f(x) = x/\log x + x/\log^2 x + 2.21x/\log^3 x$. Proposition 5 implies that $\pi(x) < f(x)$ for every $x \geq 180,250,881,352,396$. If we compare $f(x)$ with the integral logarithm $\mathrm{li}(x)$, we get by Lemma 2 that $\pi(x) < f(x)$ for every $x \geq 162,791,795,110,834$. Next, we use a computer to verify the inequality $\pi(x) < f(x)$ for every x with $x_0 \leq x \leq 162,791,795,110,834$. The remaining inequalities can be proved in the same way. □

5. Proof of Theorem 3

In order to give a proof of Theorem 3, we use (44) and a numerical calculation that verifies the desired inequality for smaller values of x.

[1]This inequality was already known to be true for every $x \geq 8 \times 10^{11}$ (see [50, Proposition 3.3]

Proof of Theorem 3. Let $x_1 = 1{,}757{,}126{,}630{,}797$. Further, let $g(x)$ be the right-hand side of (13). We can compute that $J_{3,-0.024334,x_1}(x_1) - g(x_1) > 6 \times 10^3$. In addition we have $J'_{3,-0.024334,x_1}(x) > g'(x)$ for every $x \geq 44.42$. Therefore, we get $J_{3,-0.024334,x_1}(x) > g(x)$ for every $x \geq x_1$. Using (44), we get the required inequality for every $x \geq x_1$. For smaller values of x we use a computer. □

Remark 9. Let $x_1 = 1{,}751{,}189{,}194{,}177$. Then the inequality (13) does not hold for $x = x_1 - 0.1$.

Remark 10. Theorem 3 improves the lower bound for $\pi(x)$ obtained in [5, Theorem 3].

In the next corollary, we establish some weaker lower bounds for the prime counting function.

Corollary 4. *We have*

$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{2.975666}{\log^2 x} - \frac{a_3}{\log^3 x} - \frac{a_4}{\log^4 x} - \frac{a_5}{\log^5 x}}$$

for every $x \geq x_0$, where a_3, a_4, a_5, and x_0 are given as in Table 5.

a_3	13.024334	13.024334	13.024334	0
a_4	70.951332	70.951332	0	0
a_5	460.634397856444	0	0	0
x_0	$1{,}035{,}745{,}443{,}241$	$153{,}887{,}581{,}621$	$7{,}713{,}187{,}213$	$54{,}941{,}209$

Table 5: Explicit values for a_3, a_4, a_5, and x_0.

Proof. From Theorem 3, it follows that each required inequality holds for every $x \geq 1{,}751{,}189{,}194{,}177$. For smaller values of x we use a computer. □

Let n be a positive integer. Then (48) provides the inequality

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x} + \frac{24x}{\log^5 x} + \ldots + \frac{(n-1)!x}{\log^n x}$$

for all sufficiently large values of x. In the following proposition, we describe a method to find lower bounds for $\pi(x)$ in the direction of (5) by using lower bounds for $\pi(x)$ in the direction of (11).

Proposition 6. *Let n be a positive integer and let $a_0 > 0$ and $a_1, \ldots, a_n$ be negative real numbers. Suppose that there is a positive real number x_0 such that the inequalities*

$$a_0 \log x + a_1 + \frac{a_2}{\log x} + \ldots + \frac{a_n}{\log^{n-1} x} > 0 \tag{49}$$

and

$$\pi(x) > \frac{x}{a_0 \log x + a_1 + \frac{a_2}{\log x} + \ldots + \frac{a_n}{\log^{n-1} x}} \tag{50}$$

hold simultaneously for every $x \geq x_0$. *Then we have*

$$\pi(x) > \frac{b_0 x}{\log x} + \frac{b_1 x}{\log^2 x} + \ldots + \frac{b_n x}{\log^{n+1} x}$$

for every $x \geq x_0$, *where* $b_0, \ldots, b_n$ *are real numbers recursively defined by*

$$b_0 = 1/a_0, \qquad \text{and} \qquad b_k = -\frac{1}{a_0} \sum_{i=1}^{k} a_i b_{k-1} \quad (1 \leq k \leq n). \tag{51}$$

Proof. For $y > 0$, we define $R(y) = \sum_{k=0}^{n} a_i/y^i$ and $S(y) = \sum_{i=0}^{n} b_i/y^i$. For $i \in \{1, \ldots, 2n\}$, we set

$$a_i' = \begin{cases} a_i, & \text{if } i \in \{1, \ldots, n\}, \\ 0, & \text{otherwise} \end{cases} \qquad \text{and} \qquad b_i' = \begin{cases} b_i, & \text{if } i \in \{1, \ldots, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

Using (51) together with $b_{n+1}' = 0$, we can see that

$$R(y)S(y) = 1 + \sum_{k=n+1}^{2n} \sum_{i=1}^{k} \frac{a_i' b_{k-i}'}{y^k}.$$

Since $a_i' b_{k-i}' \leq 0$ for every i with $1 \leq i \leq 2n$ and every k satisfying $n+1 \leq k \leq 2n$, we get $R(y)S(y) \leq 1$. By (49), we have $R(\log x) > 0$ for every $x \geq x_0$. Now we can use (50) to get $\pi(x) > x/(R(x)\log x) \geq xS(\log x)/\log x$ for every $x \geq x_0$. □

The best explicit result in the direction of (50) was found in [5, Proposition 5]. The following refinements of it are a consequence of Proposition 6, Theorem 3, and Corollary 4.

Corollary 5. *We have*

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{5.975666x}{\log^4 x} + \frac{b_5 x}{\log^5 x} + \frac{b_6 x}{\log^6 x} + \frac{b_7 x}{\log^7 x} + \frac{b_8 x}{\log^8 x}$$

for every $x \geq x_0$, *where* b_5, b_6, b_7, b_8, *and* x_0 *are given as in Table 6.*

b_5	b_6	b_7	b_8	x_0
23.975666	119.87833	719.26998	5034.88986	$1,681,111,802,141$
23.975666	119.87833	719.26998	0	$721,733,241,667$
23.975666	119.87833	0	0	$110,838,719,141$
23.975666	0	0	0	$1,331,691,853$
0	0	0	0	$10,383,799$

Table 6: Explicit values for a_2 and x_2.

Proof. In order to prove the first inequality, we combine Proposition 6 and Theorem 3 to see that this inequality holds for every $x \geq 1,751,189,194,177$. For smaller values of x, we use a computer. Further, we use Proposition 6, Corollary 4, and a direct computation for smaller values of x to verify the remaining inequalities. $\square$

6. Proof of Theorem 4

In order to prove Theorem 4, we set $R = 5.5666305$ and, similar to [56, p. 879], we define the function $a : \mathbb{R}_{>0} \to \mathbb{R}$ by

$$\frac{a(x)}{\log^6 x} = \begin{cases} \dfrac{2 - \log 2}{2} & \text{if } 2 \leq x < 599, \\ \dfrac{\log^2 x}{8\pi\sqrt{x}} & \text{if } 599 \leq x < 1.101 \times 10^{26}, \\ \sqrt{\dfrac{8}{17\pi}} \left(\dfrac{\log x}{6.455} \right)^{1/4} \exp\left(-\sqrt{\dfrac{\log x}{6.455}} \right) & \text{if } 1.101 \times 10^{26} \leq x < e^{673}, \\ 121.0961 \left(\dfrac{\log x}{R} \right)^{3/2} \exp\left(-2\sqrt{\dfrac{\log x}{R}} \right) & \text{if } x \geq e^{673}. \end{cases}$$

Then we get the following result concerning Chebyshev's ϑ-function.

Lemma 4. *For every $x \geq 2$, we have*

$$|\vartheta(x) - x| \leq \frac{a(x)x}{\log^6 x}.$$

Proof. In the case where x satisfies $2 \leq x < 599$, then the given bound is trivial. For second bound, see Johnston [39, Corollary 3.3]. The third bound was given by Trudgian [70, Theorem 1] and the last bound was recently established by Fiori, Kadiri, and Swidinsky [12, Corollary 14] (cf. (3)). $\square$

We also need the following result on our function a.

Lemma 5. *Let x_1 be a real number with $x_1 \geq e^{673}$. Then $a_n(x) \leq a_n(x_1)$ for every $x \geq x_1$.*

Proof. By a straightforward calculation of the derivative, we see that $a'(x) < 0$ for every $x \geq e^{673}$. $\square$

Now we use Theorem 3 and Lemmata 4-5 to give the following proof of Theorem 4.

Proof of Theorem 4. In [6, Theorem 1], the inequality was already proved for every x with $65,405,887 \leq x \leq 2.7358 \times 10^{40}$. If we utilize Theorem 3, it turns out that the inequality (14) holds unconditionally for every x such that $65,405,887 \leq x \leq e^{540}$. Now, let $f(x)$ denote the right-hand side of (14). In order to verify the required inequality for every x with $e^{540} \leq x \leq e^{1680}$, we set $c_0 = 1 - 1.6341 \times 10^{-12}$. By [34, Table 3], we have $\vartheta(x) \geq c_0 x$ for every $x > e^{500}$. Applying this inequality to (7), we get

$$\pi(x) > g_0(x) \tag{52}$$

for every $x \geq e^{500}$, where $g_0(x) = c_0(\mathrm{li}(x) - \mathrm{li}(e^{500}) + e^{500}/500)$. If we show that $g_0(x) > f(x)$ for every x satisfying $e^{540} \leq x \leq e^{1680}$, we can use (52) to see that the required inequality (14) holds for every x with $e^{540} \leq x \leq e^{1680}$. Since $g_0'(x) > f'(x)$ for every x so that $9 \leq x \leq e^{1680}$, it remains to show that $g_0(x_0) > f(x_0)$, where $x_0 = e^{540}$. First, we note that $t\sum_{k=1}^{6}(k-1)!/\log^k t < \mathrm{li}(t) < 1.003t/\log t$, where the left-hand side inequality holds for every $t \geq 565$ and the right-hand side inequality is valid for every $t \geq e^{500}$. Therefore,

$$\frac{g_0(x_0) - f(x_0)}{x_0} > c_0 \sum_{k=1}^{6} \frac{(k-1)!}{540^k} - \frac{0.003c_0}{e^{40}} - \frac{f(x_0)}{x_0}.$$

Since the right-hand side of the last inequality is positive and we conclude that the required inequality holds for every x with $x_0 \leq x \leq e^{1680}$. The final step of the proof consists in the verification of the required inequality for every $x \geq x_1$, where $x_1 = e^{1680}$. If we combine (7) with Lemma 4, we can see that

$$\pi(x) \geq \frac{x}{\log x} - \frac{xa(x)}{\log^7 x} + \int_2^x \frac{\mathrm{d}t}{\log^2 t} - \int_2^x \frac{a(t)}{\log^8 t}\,\mathrm{d}t. \tag{53}$$

Integration by parts in (53) provides that

$$\pi(x) \geq C + x\sum_{k=0}^{5} \frac{k!}{\log^{k+1} x} + \frac{(720 - a(x))x}{\log^7 x} + \int_{x_1}^x \frac{5040 - a(t)}{\log^8 t}\,\mathrm{d}t,$$

where

$$C = \int_2^{x_1} \frac{5040 - a(t)}{\log^8 t}\,\mathrm{d}t - 2\sum_{k=1}^{6} \frac{k!}{\log^{k+1} 2}.$$

Since $0 < a(t) \leq a(x_1)$ for every $t \geq x_1$ (cf. Lemma 5), it turns out that

$$\pi(x) \geq C + x\sum_{k=0}^{5} \frac{k!}{\log^{k+1} x} + \frac{(720 - a(x_1))x}{\log^7 x} + (5040 - a(x_1))\int_{x_1}^x \frac{\mathrm{d}t}{\log^8 t}.$$

Note that $5040 - a(x_1) < 0$. Hence

$$\pi(x) > C + x\sum_{k=0}^{5} \frac{k!}{\log^{k+1} x} + \frac{(720 - a(x_1))x}{\log^7 x} + (5040 - a(x_1))E(x_1),$$

where
$$E(x) = \frac{1}{5040}\left(\mathrm{li}(x) - \sum_{k=1}^{7} \frac{(k-1)!x}{\log^k x}\right).$$
Since $C + (5040 - a(x_1))E(x_1) < 0$, we obtain that
$$\pi(x) > x\sum_{k=0}^{5} \frac{k!}{\log^{k+1} x} + \frac{x}{\log^7 x}\left(720 - a(x_1) + (5040 - a(x_1))E(x_1) \times \frac{\log^7 x_1}{x_1}\right).$$
Now we use a computer to get that
$$\pi(x) > x\sum_{k=0}^{5} \frac{k!}{\log^{k+1} x} - \frac{6918930x}{\log^7 x}$$
for every $x \geq x_1$. Now we set
$$H(y) = \sum_{k=0}^{5} \frac{k!}{y^{k+1}} - \frac{6918930}{y^7} - \frac{1}{y - 1 - \frac{1}{y} - \frac{3}{y^2}}.$$
It is easy to see that $H'(y) < 0$ for every $y \geq 859$. Together with $\lim_{y\to\infty} H(y) = 0$, it turns out that $H(\log x) \geq 0$ for every $x \geq e^{859}$. If we combine the last inequality with (6), we get that $\pi(x) > f(x)$ for every $x \geq e^{1680}$ and we arrive at the end of the proof. □

Remark 11. The method employed in the proof of Theorem 4 can also be used to find further lower bounds for $\pi(x)$ given by truncating the asymptotic expansion (11) at later terms (with $\log^n x$ in the denominator, where $n \geq 3$). However, these bounds will only hold when x is exceptionally large. For instance, (11) provides the even sharper inequality
$$\pi(x) > \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{3}{\log^2 x} - \frac{13}{\log^3 x}}$$
for all sufficiently large values of x. Similar to the proof of Theorem 4, we get that this inequality holds for every x satisfying $11,471,757,461 \leq x \leq e^{57.820987}$ and every $x \geq e^{3661.424}$.

Corollary 6. *For every $x \geq 10,384,261$, we have*
$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} + \frac{2x}{\log^3 x} + \frac{6x}{\log^4 x}.$$

Proof. It suffices to combine Proposition 6, Theorem 4, and [6, Theorem 2]. □

7. Proof of Theorem 5

In this section, we want to find unrestricted effective estimates for the sum of the reciprocals of all prime numbers not exceeding x For this purpose, we use the method investigated by Rosser and Schoenfeld [63, p. 74]. They derived a remarkable identity which connects the sum of the reciprocals of all prime numbers not exceeding x with Chebyshev's ϑ-function by showing that

$$A_1(x) = \frac{\vartheta(x) - x}{x \log x} - \int_x^\infty \frac{(\vartheta(y) - y)(1 + \log y)}{y^2 \log^2 y} \, \mathrm{d}y, \tag{54}$$

where

$$A_1(x) = \sum_{p \leq x} \frac{1}{p} - \log \log x - B.$$

Here, the constant B is defined as in (16). Applying (2) to (54), Rosser and Schoenfeld [63, p. 68] refined the error term in Mertens' result (15) by giving $A_1(x) = O(\exp(-a\sqrt{\log x}))$ as $x \to \infty$, where a is an absolute positive constant. Then [63, Theorem 5] they used explicit estimates for Chebyshev's ϑ-function to show that

$$-\frac{1}{2\log^2 x} < A_1(x) < \frac{1}{2\log^2 x},$$

where the left-hand side inequality is valid for every $x > 1$ and the right-hand side inequality holds for every $x \geq 286$. Meanwhile there are several improvements of (7) (see, for instance, [30, Theorem 5.6] and [5, Proposition 7]). In Theorem 5, we give the current best unconditionally effective estimates for $A_1(x)$. The proof is now rather simple.

Proof of Theorem 5. It suffices to combine (54) with Proposition 1. □

Remark 12. Note that the positive integer $N_0 = 1,757,126,630,797$ might not be the smallest positive integer N so that the inequality given in Theorem 5 holds for every $x \geq N$.

Remark 13. Rosser and Schoenfeld [63, Theorem 20] used the calculation in [1] to see that $A_1(x) > 0$ for every $1 < x \leq 10^8$ and raised the question whether this inequality hold for every $x > 1$. Robin [62, Théorème 2] proved that the function $A_1(x)$ changes the sign infinitely often, which leads to a negative answer to the obove question. By adapting a method for bounding Skewes' number, Büthe [13, Theorem 1.1] found that there exists an $x_0 \in [\exp(495.702833109), \exp(495.702833165)]$ such that $A_1(x)$ is negative for every $x \in [x_0 - \exp(239.046541), x_0]$.

Remark 14. Under the assumption that the Riemann hypothesis is true, Schoenfeld [66, Corollary 2] found some better estimate for the sum of the reciprocals of all prime numbers not exceeding x. This result was recently improved by Dusart [31, Theorem 4.1].

Using the definition (16) of B, we get

$$e^{\gamma} \log x \prod_{p \leq x} \left(1 - \frac{1}{p}\right) = e^{-S(x) - A_1(x)}, \tag{55}$$

where

$$S(x) = \sum_{p > x} \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right) = -\sum_{n=2}^{\infty} \frac{1}{n} \sum_{p > x} \frac{1}{p^n}. \tag{56}$$

By Rosser and Schoenfeld [63, p. 87], we have

$$-\frac{1.02}{(x-1)\log x} < S(x) < 0 \tag{57}$$

for every $x > 1$. Hence, the asymptotic formula (15) gives $A_2(x) = O(1/\log^2 x)$ as $x \to \infty$, where

$$A_2(x) = \frac{e^{-\gamma}}{\log x} - \prod_{p \leq x} \left(1 - \frac{1}{p}\right).$$

In [63, Theorem 7], Rosser and Schoenfeld found that

$$\frac{e^{-\gamma}}{\log x} \left(1 - \frac{1}{2\log^2 x}\right) < \prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x} \left(1 + \frac{1}{2\log^2 x}\right),$$

where the left-hand side inequality is valid for every $x \geq 285$ and the right-hand side inequality holds for every $x > 1$. We use (55) combined with Theorem 5 to obtain the following refinement of [5, Proposition 9].

Proposition 7. *For every $x \geq 1,757,126,630,797$, we have*

$$\frac{e^{-\gamma}}{\log x} \exp(-f(x)) < \prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x} \exp\left(f(x) + \frac{1.02}{(x-1)\log x}\right),$$

where $f(x)$ denotes the right-hand side of (17).

Proof. First, we apply the left-hand side inequality of Theorem 5 to (55) and see that

$$\prod_{p \leq x} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x} \exp(-S(x) + f(x)) \tag{58}$$

for every $x > 1,757,126,630,797$. Now it suffices to apply the right-hand side inequality of (57) to (58) and we get the required right-hand side inequality. One the other hand, we have $S(x) < 0$ by (57). Applying this and (17) to (55), we arrive at the end of the proof. $\square$

Remark 15. Note that the positive integer $N_0 = 1,757,126,630,797$ in Proposition 7 might not be the smallest positive integer N so that the inequality given holds for every $x \geq N$.

Remark 16. Under the assumption that the Riemann hypothesis is true, Schoenfeld [66, Corollary 3] found that the inequality

$$|A_2(x)| < \frac{3\log x + 5}{8\pi e^{\gamma}\sqrt{x}\log x}$$

holds for every $x \geq 8$. This was slightly improved by Dusart [31, Theorem 4.4] in 2018.

Remark 17. Rosser and Schoenfeld [63, Theorem 23] found that $A_2(x) > 0$ for every $0 < x \leq 10^8$ and stated [63, p. 73] the question whether this inequality also hold for every $x > 10^8$. In [62, Proposition 1], Robin answered this by showing that the function $A_2(x)$ changes the sign infinitely often.

Now we can use Proposition 7 to derive the following effective estimates for

$$\prod_{p \leq x}\left(1 + \frac{1}{p}\right),$$

where p runs over primes not exceeding x.

Corollary 7. *For every $x \geq 1,757,126,630,797$, one has*

$$\begin{aligned}\frac{6e^{\gamma}}{\pi^2}\exp\left(-f(x) - \frac{1.02}{(x-1)\log x}\right)\log x &< \prod_{p \leq x}\left(1 + \frac{1}{p}\right) \\ &< \frac{6e^{\gamma}}{\pi^2}\left(1 + \frac{1}{x}\right)\exp(f(x))\log x,\end{aligned}$$

where $f(x)$ denotes the right-hand side of (17).

Proof. Since $1 + 1/p = (1 - 1/p^2)/(1 - 1/p)$ and $\zeta(2) = \pi^2/6$, it suffices to combine Proposition 7 and [31, Lemma 4.3]. □

Remark 18. Note that the positive integer $N_0 = 1,757,126,630,797$ might not be the smallest positive integer N so that the inequality given in Corollary 7 holds for every $x \geq N$.

Let us briefly study $S(x)$, defined as in (56), in more detail. In the proof of the left-hand side inequality in (57), Rosser and Schoenfeld used the inequality $\vartheta(x) < 1.02x$ which is valid for every $x > 0$ (see [63, Theorem 9]). If we use approximations for $\vartheta(x)$ of the form (5), we get the following result.

Proposition 8. *Let k be a positive integer and let η_k and $x_0 = x_0(k)$ be positive real numbers with $x_0 > 1$ so that $|\vartheta(x) - x| < \eta_k x/\log^k x$ for every $x \geq x_0$. Then, we have*

$$\left| S(x) - \sum_{n=1}^{\infty} \frac{\mathrm{li}(x^{-n})}{n+1} \right| < \frac{\eta_k}{\log^{k+1} x} \left((x+1) \log \left(\frac{x}{x-1} \right) - 1 \right)$$

for every $x \geq x_0$.

In order to prove this proposition, we first establish the following lemma.

Lemma 6. *Let n be a positive integer with $n \geq 2$. Under the assumptions of Proposition 8, we have*

$$\left| \mathrm{li}(x^{1-n}) + \sum_{p>x} \frac{1}{p^n} \right| < \frac{\eta_k}{x^{n-1} \log^{k+1} x} \left(1 + \frac{n}{n-1} \right)$$

for every $x \geq x_0$.

Proof. By [63, p. 87], we have

$$\sum_{p>x} \frac{1}{p^n} = -\frac{\vartheta(x)}{x^n \log x} + \int_x^{\infty} \frac{(1 + n \log y)\vartheta(y)}{y^{n+1} \log^2 y} \, \mathrm{d}y. \tag{59}$$

Since we have assumed that $|\vartheta(x) - x| < \eta_k x/\log^k x$ for every $x \geq x_0$, we see that

$$\sum_{p>x} \frac{1}{p^n} \leq -\mathrm{li}(x^{1-n}) + \frac{\eta_k}{x^{n-1} \log^{k+1} x} + \eta_k \int_x^{\infty} \frac{1 + n \log y}{y^n \log^{k+2} y} \, \mathrm{d}y \tag{60}$$

for every $x \geq x_0$. Analogous to [63, Lemma 9], we get that

$$\int_x^{\infty} \frac{1 + n \log y}{y^n \log^{k+2} y} \, \mathrm{d}y \leq \frac{n}{(n-1)x^{n-1} \log^{k+1} x}.$$

Applying this inequality to (60), we see that the required upper bound holds for every $x \geq x_0$. The proof of the required lower bound is quite similar and we leave the details to the reader. □

Now we can combine the definition (56) with Lemma 6 to get the following proof of Proposition 8.

Proof of Proposition 8. If we apply Lemma 6 to (56), it turns out that

$$\left| S(x) - \sum_{n=1}^{\infty} \frac{\mathrm{li}(x^{-n})}{n+1} \right| < \frac{\eta_k}{\log^{k+1} x} \sum_{n=2}^{\infty} \left(1 + \frac{n}{n-1} \right) \frac{1}{n x^{n-1}}$$

for every $x \geq x_0$. Now, it suffices to apply the identity

$$\sum_{n=2}^{\infty} \left(1 + \frac{n}{n-1}\right) \frac{1}{nx^{n-1}} = (x+1) \log\left(\frac{x}{x-1}\right) - 1$$

to complete the proof. □

If we combine (35) and (59), we find the following new necessary condition for the Riemann hypothesis including the sum given in Lemma 6.

Proposition 9. *Let n be a positive integer with $n \geq 2$. Under the assumption that the Riemann hypothesis is true, we have*

$$\left| \mathrm{li}(x^{1-n}) + \sum_{p>x} \frac{1}{p^n} \right| < \frac{1}{8\pi x^{n-1/2}} \left(1 + \frac{2n}{2n-1}\right) \left(\log x + \frac{2}{2n-1}\right)$$

for every $x \geq 599$.

Proof. Instead of the assumption (5), we now use (35) in the proof of Lemma 6. □

8. Proof of Theorem 6

Here we give the following proof of Theorem 6.

Proof of Theorem 6. Let the constant E be defined as in (19) and let

$$A_3(x) = \sum_{p \leq x} \frac{\log p}{p} - \log x - E.$$

By Rosser and Schoenfeld [63, p. 74], we have

$$A_3(x) = \frac{\vartheta(x) - x}{x} - \int_x^{\infty} \frac{\vartheta(y) - y}{y^2}\, \mathrm{d}y. \tag{61}$$

Similarly to the proof of Theorem 5, we may combine (61) and Proposition 1 to get that the desired both inequalities hold for every $x \geq 1{,}757{,}126{,}630{,}797$. □

Remark 19. Under the assumption that the Riemann hypothesis is true, Schoenfeld [66, Corollary 2] found a better upper bound for $|A_3(x)|$. This result was later improved by Dusart [31, Theorem 4.2].

Remark 20. Rosser and Schoenfeld [63, Theorem 21] also found that $A_3(x) > 0$ for every $0 < x \leq 10^8$. Again, they asked whether this inequality also holds for every $x > 10^8$. Robin [62, Proposition 1] showed that the function $A_3(x)$ changes the sign infinitely often, which leads again to a negative answer to the above question. Unfortunately, until today no x_0 is known so that $A_3(x_0) < 0$.

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Appendix

Here we use Corollary 1 and Walisch's *primecount* program [74] to note more weaker upper bounds for $\pi(x)$ of the form (4), where m is an integer with $0 \leq m \leq 2$ and $a_0, \ldots, a_m$ are suitable positive real numbers. We start with the case where $m = 0$.

Proposition 10. *One has*

$$\pi(x) < \frac{x}{\log x - a_0}$$

for every $x \geq x_0$, where a_0 and x_0 are given as in Table 7 and Table 8.

a_0	1.0344	1.0345	1.0346
x_0	98,011,218,006,714	90,093,726,828,053	82,972,765,680,514
a_0	1.0347	1.0348	1.0349
x_0	76,292,362,570,940	70,363,470,737,452	64,716,191,738,353
a_0	1.035	1.036	1.037
x_0	59,667,044,596,151	27,086,141,056,455	12,806,615,320,917
a_0	1.038	1.039	1.04
x_0	6,317,261,904,937	3,231,501,496,562	1,697,021,254,855
a_0	1.041	1.042	1.043
x_0	924,640,658,874	519,205,451,664	296,735,291,225
a_0	1.044	1.045	1.046
x_0	175,758,684,156	105,640,136,371	65,431,161,562
a_0	1.047	1.048	1.049
x_0	41,022,022,044	25,724,702,310	17,231,171,472
a_0	1.05	1.051	1.052
x_0	11,207,440,881	7,538,561,672	5,047,295,951
a_0	1.053	1.054	1.055
x_0	3,745,835,388	2,605,443,747	1,810,796,757
a_0	1.056	1.057	1.058
x_0	1,220,594,340	876,542,559	673,828,570
a_0	1.059	1.06	1.061
x_0	501,155,566	383,446,375	269,585,283
a_0	1.062	1.063	1.064
x_0	196,894,353	180,220,137	116,749,925
a_0	1.065	1.066	1.067
x_0	110,166,540	76,223,058	53,431,171
a_0	1.068	1.069	1.07
x_0	46,097,944	39,706,453	31,027,247

Table 7: Explicit values for a_0 and x_0.

a_0	1.071	1.072	1.073	1.074	1.075
x_0	22,078,017	18,339,738	13,026,859	12,895,928	8,832,927
a_0	1.076	1.077	1.078	1.079	1.08
x_0	7,299,254	7,117,256	5,465,656	4,994,010	3,462,478
a_0	1.081	1.082	1.083	1.08366	1.084
x_0	3,455,648	2,279,177	1,529,630	1,526,671	1,525,432
a_0	1.085	1.086	1.087	1.088	1.089
x_0	1,515,074	1,200,014	1,195,296	624,878	618,726
a_0	1.09	1.091	1.092	1.093	1.094
x_0	618,058	445,112	359,804	356203	355,990
a_0	1.095	1.096	1.097	1.098	1.099
x_0	355,177	155,935	155,907	60,297	60,224

Table 8: Explicit values for a_0 and x_0.

Proof. Let $f(x) = x/(\log x - 1.0344)$. Corollary 1 implies that

$$\pi(x) < \frac{x}{\log x - 1.0344}$$

for every $x \geq 106,640,139,304,611$. If we compare the right-hand side of (8) with the integral logarithm $\mathrm{li}(x)$, we can use Lemma 2 to see that the inequality (8) also holds for every x with $98,269,667,551,459 \leq x \leq 106,640,139,304,611$. We conclude by direct computation. $\square$

Remark 21. The real number $a_0 = 1.08366$ in Proposition 10 is mostly only of historical value. On the basis of his study of a limited table of primes, Legendre stated 1808 (see [46, p. 394]) that $\pi(x) = x/(\log x - A(x))$, where $\lim_{x\to\infty} A(x) = 1.08366$. Clearly Legendre's conjecture is equivalent to (8). However, from (11), it follows that the best value of $\lim_{x\to\infty} A(x)$ is 1. At this point it should be mentioned that Panaitopol [52] claimed to have proved the inequality

$$\pi(x) < \frac{x}{\log x - 1.08366} \tag{62}$$

for every $x > 10^6$. In Proposition 10, it could be shown that $N = 1,526,671$ is the smallest possible positive integer so that the inequality (62) holds for every $x \geq N$.

Next, we obtain the following effective estimates for $\pi(x)$ for the case where $m = 1$. The proof is similar to the proof of Proposition 10 and is left to the reader.

Proposition 11. *We have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{a_1}{\log x}}$$

for every $x \geq x_1$, where a_1 and x_1 are given as in Table 9.

a_1	1.11	1.1105	1.111
x_1	62,998,850,942,976	55,193,608,062,217	49,246,036,992,716
a_1	1.112	1.113	1.114
x_1	38,472,138,880,411	30,658,643,813,468	23,767,640,743,883
a_1	1.115	1.116	1.117
x_1	19,278,513,358,342	15,142,627,022,527	12,279,648,138,508
a_1	1.118	1.119	1.12
x_1	9,684,114,630,824	7,981,446,192,206	6,323,967,140,812
a_1	1.121	1.122	1.123
x_1	5,273,225,700,761	4,170,462,893,841	3,458,549,136,539
a_1	1.124	1.125	1.126
x_1	2,825,539,807,244	2,292,448,124,593	1,903,596,231,542
a_1	1.127	1.128	1.129
x_1	1,573,767,234,188	1,290,096,268,844	1,073,403,839,693
a_1	1.13	1.131	1.132
x_1	889,377,392,161	782,989,678,664	608,408,258,090
a_1	1.133	1.134	1.135
x_1	540,050,850,157	452,875,824,702	373,479,021,700
a_1	1.136	1.137	1.138
x_1	335,562,521,091	263,728,502,964	242,118,904,367
a_1	1.139	1.14	1.141
x_1	201,924,836,111	161,054,192,492	149,061,190,565
a_1	1.142	1.143	1.144
x_1	125,233,112,846	105,053,836,224	86,061,321,374
a_1	1.145	1.146	1.147
x_1	77,278,924,451	61,344,524,412	57,720,831,343
a_1	1.148	1.149	1.15
x_1	46,039,922,948	42,575,222,481	38,284,442,297

Table 9: Explicit values for a_1 and x_1.

Finally, we consider the case where $m = 2$ and find the following explicit estimates for $\pi(x)$. Again, the proof is quite similar to the proof of Proposition 10 and we leave the details to the reader.

Proposition 12. *We have*

$$\pi(x) < \frac{x}{\log x - 1 - \frac{1}{\log x} - \frac{a_2}{\log^2 x}}$$

for every $x \geq x_2$, where a_2 and x_2 are given as in Table 10.

a_2	3.49	3.495	3.5
x_2	$83,027,761,686,134$	$63,024,307,127,421$	$50,794,512,296,846$
a_2	3.51	3.52	3.53
x_2	$30,594,003,254,258$	$17,348,455,129,950$	$11,655,963,556,138$
a_2	3.54	3.55	3.56
x_2	$5,539,984,798,515$	$4,489,052,430,063$	$2,180,930,569,481$
a_2	3.57	3.58	3.59
x_2	$1,464,200,206,021$	$882,055,689,961$	$584,256,118,105$
a_2	3.6	3.61	3.62
x_2	$437,882,804,654$	$332,203,763,508$	$201,890,631,296$
a_2	3.63	3.64	3.65
x_2	$148,632,348,138$	$102,965,110,268$	$55,102,251,180$
a_2	3.66	3.67	3.68
x_2	$38,278,086,931$	$24,178,954,639$	$21,729,109,565$

Table 10: Explicit values for a_2 and x_2.