#A37 INTEGERS 24 (2024)

LOWER RATIONAL APPROXIMATIONS AND FAREY STAIRCASES

David Harry Richman
Department of Mathematics, University of Washington, Seattle, Washington
hrichman@uw.edu

Received: 11/30/23, Revised: 1/23/24, Accepted: 3/16/24, Published: 4/8/24

Abstract

For a real number x, call $\frac{1}{n}\lfloor nx \rfloor$ the n-th lower rational approximation of x. We study the functions defined by taking the cumulative average of the first n lower rational approximations of x, which we call the Farey staircase functions. This sequence of functions is monotonically increasing. We determine limit behavior of these functions and show that they exhibit fractal structure under appropriate normalization.

1. Introduction

Let $\lfloor x \rfloor$ denote the greatest integer no larger than x. The function $x \mapsto \lfloor x \rfloor$ is commonly called the *floor function.* Its graph looks like a staircase consisting of unit-height steps at each integer; we could also call $x \mapsto \lfloor x \rfloor$ the *unit staircase function.*

For a positive integer n, the function $x \mapsto \frac{1}{n}\lfloor nx \rfloor$ is a rescaled staircase function. The rescaled staircase still has "unit slope" between steps, but each step now has size $1/n$. The sequence of functions $\{\frac{1}{n}\lfloor nx \rfloor : n = 1, 2, \ldots\}$ approaches the identity function x from below, but the convergence is not monotonic: it is not generally true that $\frac{1}{m}\lfloor mx \rfloor \leq \frac{1}{n}\lfloor nx \rfloor$ if $m \leq n$.

Note that $\frac{1}{n}\lfloor nx \rfloor$ is the largest element of $\frac{1}{n}\mathbb{Z}$ which is no larger than x; in symbols,

$$\tfrac{1}{n}\lfloor nx \rfloor = \max\{y \in \tfrac{1}{n}\mathbb{Z} : y \leq x\}, \tag{1}$$

which motivates us to call $\frac{1}{n}\lfloor nx \rfloor$ the *n-th lower rational approximation* of x.

Consider taking the cumulative average of the first n lower rational approximations,

$$A_n(x) := \frac{1}{n}\Big(\lfloor x \rfloor + \tfrac{1}{2}\lfloor 2x \rfloor + \tfrac{1}{3}\lfloor 3x \rfloor + \cdots + \tfrac{1}{n}\lfloor nx \rfloor \Big). \tag{2}$$

DOI: 10.5281/zenodo.10944039

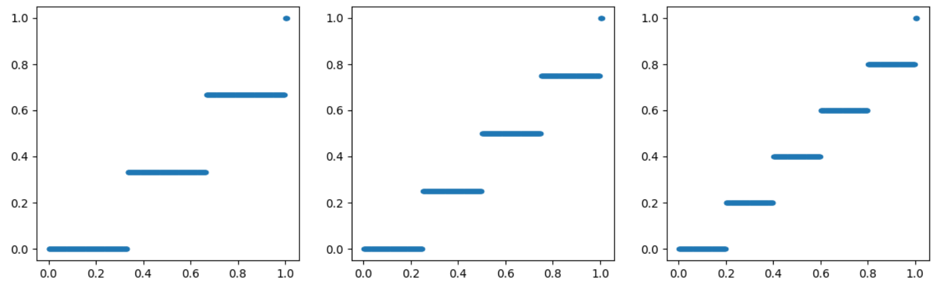

Figure 1: Lower rational approximations $\frac{1}{n}\lfloor nx \rfloor$, for $n = 3, 4, 5$.

Recall that the *Farey fractions* of order n are all fractions whose denominator have size at most n:

$$\mathcal{F}_n = \left\{ \frac{p}{q} : p \in \mathbb{Z},\ q \in \mathbb{N},\ q \leq n \right\}.$$

The function $A_n(x)$ is a step function with jump discontinuities at the Farey fractions of order n. We thus call the graph of $A_n(x)$ the *Farey staircase* of order n. See Figure 2 for small examples, and Figure 4 for a larger one.

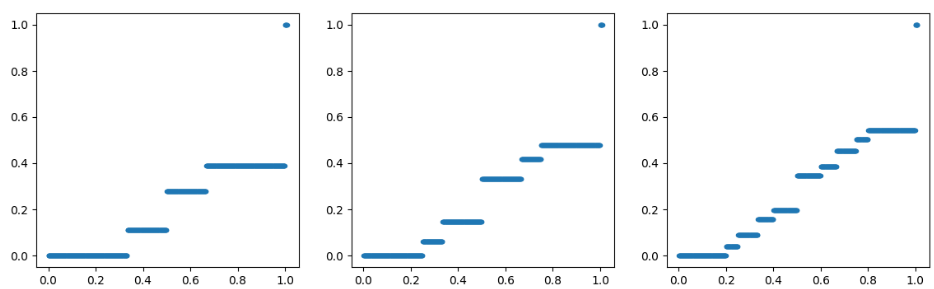

Figure 2: Farey staircases $A_n(x)$ for $n = 3, 4, 5$.

A surprising property of the Farey staircase functions is that they are monotonically increasing,

$$A_1(x) \leq A_2(x) \leq A_3(x) \leq \cdots ,$$

their values approaching x from below. This is proved in [5, 8, 11]; see also Section 1.2. In other words, if we define the incremented staircase function

$$D_n(x) = A_n(x) - A_{n-1}(x), \tag{3}$$

then $D_n(x) \geq 0$ for all x and all $n \geq 2$. See Figure 3 for small examples, and Figures 5 and 7 for larger ones.

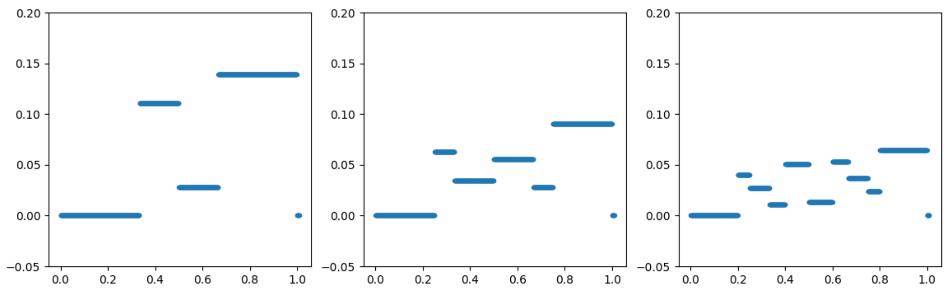

Figure 3: Incremented staircases $D_n(x)$ for $n = 3, 4, 5$.

The purpose of this paper is to study the fractal behavior of the Farey staircases $A_n(x)$ and the incremented staircases $D_n(x)$ that arises in the limit $n \to \infty$. Hints of this fractal behavior are apparent in Figures 4 and 5. We will prove theorems which quantify aspects of this fractal-like behavior.

As a consequence, we obtain a novel perspective on why we should "expect" that $D_n(x)$ takes nonnegative values (under appropriate scaling) in the limit $n \to \infty$, which does not make claims about $D_n(x)$ for individual n.

1.1. Results

To capture the fractal behavior of the Farey staircase, we first study the following rescaling of $A_n(x)$. Let

$$B_n(x) := n\, A_n(\tfrac{1}{n}x) = \left\lfloor \tfrac{1}{n}x \right\rfloor + \tfrac{1}{2}\left\lfloor \tfrac{2}{n}x \right\rfloor + \tfrac{1}{3}\left\lfloor \tfrac{3}{n}x \right\rfloor + \cdots + \tfrac{1}{n}\lfloor x \rfloor . \tag{4}$$

We obtain B_n from A_n by "zooming in" at the origin by a factor of n. In the limit $n \to \infty$, the sequence of functions B_n converges to a pointwise limit.

Theorem 1. *Suppose $x \geq 0$. As $n \to \infty$,*

$$\lim_{n\to\infty} B_n(x) = \sum_{k=1}^{\lfloor x \rfloor} \log(x/k). \tag{5}$$

For a graph of the limit function, see the left side of Figure 6.

Remark 2. We note the following observations.

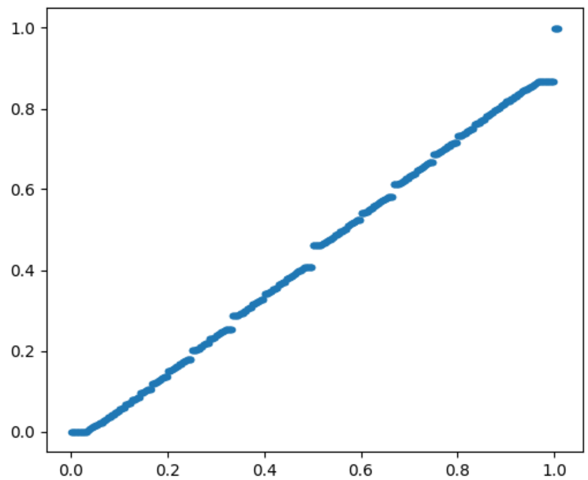

Figure 4: Farey staircase $A_{30}(x)$ on the domain $[0, 1]$.

(i) Let $B(x) \coloneqq \lim_{n\to\infty} B_n(x)$ denote the limit function in (5). Then $B(x)$ can be characterized as

$$B(x) = \sum_{k=1}^{\lfloor x \rfloor} \left(\int_k^x \frac{1}{t}\, dt \right) = \int_0^x \frac{\lfloor t \rfloor}{t}\, dt. \tag{6}$$

Thus $B(x)$ is continuous on the domain $x \geq 0$, and we have the bound

$$B(x) \;\leq\; \int_0^x \mathbb{1}(t \geq 1)\, dt \;=\; x - 1 \;\leq\; \lfloor x \rfloor\,. \tag{7}$$

Note that, in particular, $B(x) = 0$ when $0 \leq x \leq 1$.

(ii) If we let $\log^+(x) \coloneqq \max\{0, \log x\}$, then

$$B(x) = \sum_{k=1}^{\lfloor x \rfloor} \log(x/k) = \sum_{k=1}^{\infty} \log^+(x/k).$$

Since $\log^+(x)$ is continuous on the domain $x \geq 0$, this also shows that $B(x)$ is continuous.

(iii) The limit function $B(x)$ can also be expressed as $\lfloor x \rfloor \log x - \log(\lfloor x \rfloor !)$. In other words,

$$\lim_{n\to\infty} B_n(x) = \begin{cases} 0 & \text{if } 0 \le x < 1, \\ \log x & \text{if } 1 \le x < 2, \\ 2\log x - \log 2 & \text{if } 2 \le x < 3, \\ \quad\vdots & \\ k\log x - \log k! & \text{if } k \le x < k+1. \end{cases}$$

By using Stirling's approximation for $\log k!$, the bound (7) can be refined to the asymptotic

$$B(x) = x - \frac{1}{2}\log x + O(1) \qquad \text{as } x \to \infty.$$

Theorem 1 captures the behavior of the Farey staircase $A_n(x)$ in a small neighborhood of the origin. The following theorem more generally describes the limiting behavior of $A_n(x)$ in a small neighborhood above any rational point $x = \frac{p}{q}$.

Theorem 3. *Suppose $x \ge 0$. For any positive reduced fraction $\frac{p}{q}$ (i.e., p and q are positive integers with $\gcd(p,q) = 1$), we have*

$$\lim_{n\to\infty} \Big(B_n(x + \tfrac{p}{q}n) - B_n(\tfrac{p}{q}n) \Big) = \frac{1}{q} \sum_{k=1}^{\lfloor qx \rfloor} \log(qx/k).$$

In other words, the limit function here is a rescaling of the limit function $B(x)$ in (5), namely

$$\lim_{n\to\infty} \Big(B_n(x + \tfrac{p}{q}n) - B_n(\tfrac{p}{q}n) \Big) = \frac{1}{q} B(qx).$$

The scaling factor q is independent of the numerator p. By Remark 2, the limit function is equal to

$$\frac{1}{q}B(qx) \;=\; \frac{1}{q}\int_0^{qx} \frac{\lfloor t \rfloor}{t}\,dt \;=\; \int_0^{x} \frac{\lfloor qs \rfloor}{qs} ds \;=\; \frac{1}{q}\sum_{k=1}^{\infty} \log^+ \left(\frac{qx}{k}\right).$$

Next we investigate fractal behavior that arises from taking the difference of consecutive Farey staircases. As before, let

$$D_n(x) \coloneqq A_n(x) - A_{n-1}(x)$$

for $n \ge 2$, and let $D_n^{\max} = \max\{D_n(x) : x \in \mathbb{R}\}$. Figure 5 shows the graph of $D_{30}(x)/D_{30}^{\max}$. Note that Figure 5 shows a pattern of "negatively-sloped stripes." The phenomenon of these segments persists in the limit $n \to \infty$, and is explained by the following result.

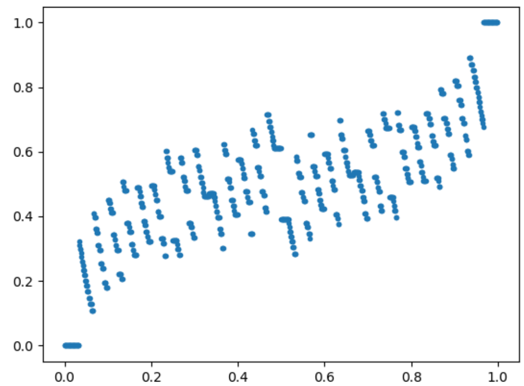

Figure 5: Incremented staircase $D_{30}(x)$, normalized to height one.

Theorem 4. *Suppose $x \geq 0$. As $n \to \infty$,*

$$\lim_{n\to\infty} n^2 D_n\left(\tfrac{1}{n}x\right) = \sum_{k=1}^{\lfloor x \rfloor} \left(1 - \log(\tfrac{x}{k})\right). \tag{8}$$

The limit function is in fact $\lfloor x \rfloor - B(x)$, where $B(x)$ is the limit function from Theorem 1. For a graph of the limit function, see the right side of Figure 6. In particular, the limit value of $n^2 D_n\left(\tfrac{1}{n}x\right)$ is nonnegative as a consequence of equation (7).

By Stirling's approximation, we have

$$\lim_{n\to\infty} n^2 D_n\left(\frac{1}{n}x\right) = \lfloor x \rfloor - B(x) = \frac{1}{2}\log x + O(1) \qquad \text{as } x \to \infty.$$

1.2. Olympiad Problem

Here we describe the initial motivation for this work.

Problem 5 of the 1981 U.S.A. Mathematical Olympiad was to prove that

$$\lfloor nx \rfloor \geq \lfloor x \rfloor + \tfrac{1}{2}\lfloor 2x \rfloor + \tfrac{1}{3}\lfloor 3x \rfloor + \cdots + \tfrac{1}{n}\lfloor nx \rfloor \tag{9}$$

where x is a real number, n is a positive integer, and $\lfloor t \rfloor$ denotes the greatest integer less than or equal to t. Solutions can be found in [5, 8, 11], and in Appendix A of this work. With some algebraic manipulation, the Olympiad inequality (9) is equivalent to the condition $A_{n-1}(x) \leq A_n(x)$ on Farey staircases.

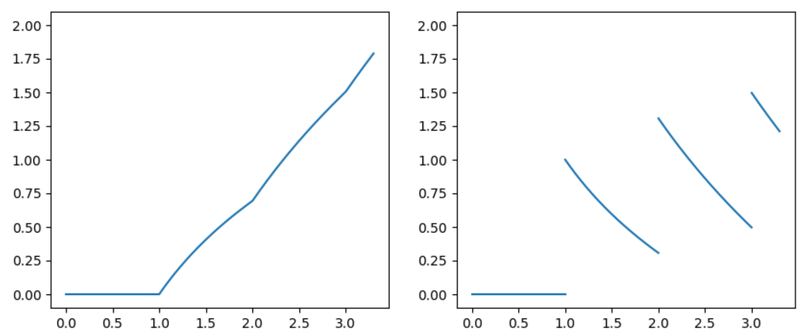

Figure 6: Limit functions $B(x) = \sum_{k=1}^{\lfloor x \rfloor} \log(x/k)$, left, and $\lfloor x \rfloor - B(x)$, right.

Let

$$f_n(x) := \lfloor nx \rfloor - \left(\lfloor x \rfloor + \tfrac{1}{2} \lfloor 2x \rfloor + \tfrac{1}{3} \lfloor 3x \rfloor + \cdots + \tfrac{1}{n} \lfloor nx \rfloor \right),$$

and let S denote the set values taken by f_n for all n,

$$S = \{ f_n(x) : x \in \mathbb{R},\ n = 1, 2, \ldots \}.$$

The Olympiad inequality states that S does not contain negative values. The following result of D. R. Richman [11] gives a more complete description of S.

Theorem 5 (see [11, Theorem 1.1]). *Let* $\lambda = 1 - \log 2$.

(i) The set S *is dense in the interval* $[\lambda, +\infty)$.

(ii) The intersection $S \cap (-\infty, \lambda - \epsilon]$ *has finitely many elements for any* $\epsilon > 0$.

Our motivation was to understand the appearance of the curious constant $1 - \log 2$ in the structure of S. Note that

$$f_n\left(\frac{1}{n} x\right) = \lfloor x \rfloor - B_n(x).$$

The main result of this paper, Theorem 1, characterizes the values of $B_n(x)$ in the limit $n \to \infty$, which allows us to understand the structure of the set S through this connection to f_n. The limiting values of $\lfloor x \rfloor - B_n(x)$ as $n \to \infty$ are shown in the right-hand side of Figure 6. The structure apparent in this figure implies the following corollary, which covers half of Theorem 5.

Corollary 6 (to Theorem 1). *The set* S *is dense in* $[1 - \log 2, +\infty)$.

To elaborate more, the graph of $\lim_{n\to\infty} (\lfloor x \rfloor - B_n(x))$ consists of continuous segments separated by jump discontinuities. The second continuous segment spans

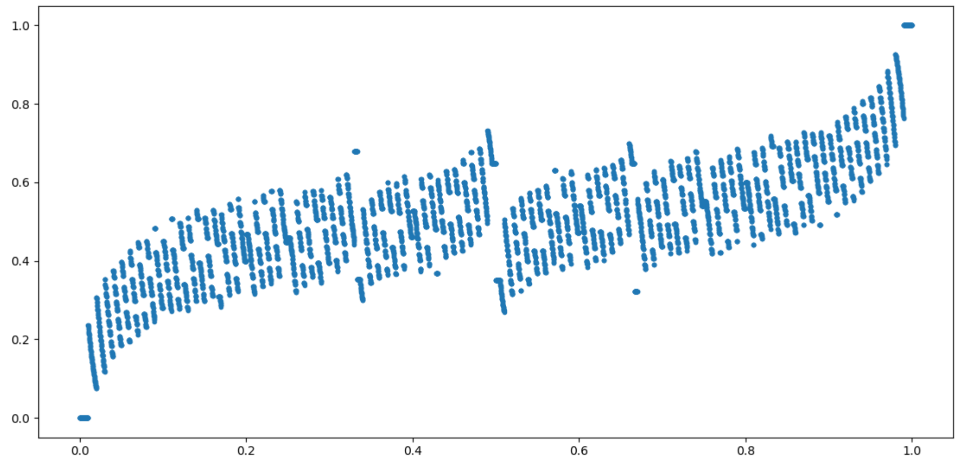

Figure 7: Incremented staircase $D_{100}(x)$, normalized to height one.

y-values between 1 and $1 - \log 2$. Each following segment overlaps in its y-values with the previous segment's y-values. The graph does not cross the region with y-values between 0 and $1 - \log 2$.

1.3. Related Work

Interesting problems concerning combinations of floor functions were posed by Ramanujan [10] and further generalized by Somu and Kukla [12]. Another useful floor function identity is attributed to Hermite, with recent generalizations given by Aursukaree, Khemaratchatakumthorn, and Pongsriiam [1]. Other sums of scaled floor function are considered by Thanatipanonda and Wong [13].

Niederreiter [9] and Dress [2] proved bounds on the discrepancy of the Farey sequence, which concerns the spacing between consecutive fractions. Kanemitsu and Yoshimoto [4] connect the Riemann hypothesis to certain estimates on sums of Farey fractions. Lagarias and Mehta [7] relate the Riemann hypothesis to properties of the product of Farey fractions of a given order. Kunik [6] also studies a family of real-valued functions derived from Farey fractions, and their limiting behavior.

1.4. Organization

This paper is organized as follows. In Section 2 we derive some basic identities satisfied by the Farey staircase functions $A_n(x)$ and their incremented functions

$D_n(x)$. In Section 3 we prove Theorems 1, 3, and 4. In Section 4 we give a heuristic argument for Theorem 1 that provides an alternative perspective. Finally in Appendix A we include a proof of the Olympiad inequality (9).

2. Preliminaries

Let $\mathcal{F}_n$ denote the Farey fractions of order n,

$$\mathcal{F}_n \coloneqq \bigcup_{m=1}^{n} \frac{1}{m}\mathbb{Z},$$

and let $\mathcal{F}_n^{[0,1]} = \mathcal{F}_n \cap [0,1]$ denote the Farey fractions on the unit interval. Recall that the *Farey staircase function* of order n is

$$A_n(x) = \frac{1}{n}\sum_{k=1}^{n} \frac{1}{k}\lfloor kx \rfloor$$

and the *incremented staircase function* of order n is

$$D_n(x) = A_n(x) - A_{n-1}(x).$$

The following identities are straightforward to verify.

Proposition 7. *The functions $A_n(x)$ and $D_n(x)$ satisfy the following, for $n \geq 2$:*

(a) $D_n(x) = \dfrac{1}{n}\left(\dfrac{1}{n}\lfloor nx \rfloor - A_{n-1}(x)\right);$

(b) $D_n(x) = \dfrac{1}{n-1}\left(\dfrac{1}{n}\lfloor nx \rfloor - A_n(x)\right).$

Proof. Note that

$$nA_n(x) = \sum_{k=1}^{n} \frac{1}{k}\lfloor kx \rfloor = \frac{1}{n}\lfloor nx \rfloor + (n-1)A_{n-1}(x). \tag{10}$$

Solving the above equation for $A_n(x)$ and subtracting $A_{n-1}(x)$, we obtain, for $n \geq 2$,

$$\begin{aligned} A_n(x) - A_{n-1}(x) &= \frac{1}{n}\left(\frac{1}{n}\lfloor nx \rfloor + (n-1)A_{n-1}(x)\right) - A_{n-1}(x) \\ &= \frac{1}{n}\left(\frac{1}{n}\lfloor nx \rfloor - A_{n-1}(x)\right), \end{aligned}$$

as claimed in (a). If we instead use (10) to solve for $A_{n-1}(x)$, then subtracting from $A_n(x)$, we obtain for $n \geq 2$ that

$$\begin{aligned} A_n(x) - A_{n-1}(x) &= A_n(x) - \frac{1}{n-1}\left(-\frac{1}{n}\lfloor nx \rfloor + nA_n(x)\right) \\ &= \frac{1}{n-1}\left(\frac{1}{n}\lfloor nx \rfloor - A_n(x)\right), \end{aligned}$$

as claimed in (b). □

Lemma 8.

(a) For $n \geq 1$, we have $A_n(x) \leq \frac{1}{n}\lfloor nx \rfloor \leq A_n(x) + \frac{\log n}{n}$.

(b) For $n \geq 2$, we have $0 \leq D_n(x) \leq \frac{\log n}{n(n-1)}$.

Proof. (a) The lower bound is equivalent to the USAMO inequality (9); a proof can be found in [5], [8], or [11, Lemma 2.1]. The upper bound follows from [11, Theorem 6.1], along with the harmonic sum bound $\sum_{k=2}^{n} \frac{1}{k} \leq \log n$.

(b) Combine Proposition 7 (b) with part (a) of this lemma. □

The following lemma allows us to interchange a sum for an integral.

Lemma 9. *Suppose $\{a_n\}$ and $\{b_n\}$ are positive integer sequences which satisfy*

- *$a_n \leq b_n$, and*
- *$a_n \to \infty$ and $b_n \to \infty$ as $n \to \infty$.*

Then $\lim_{n\to\infty} \sum_{k=a_n}^{b_n} \frac{1}{k} = \lim_{n\to\infty} \int_{a_n}^{b_n} \frac{dt}{t}$.

Proof. By consideration of Riemann sums,

$$\sum_{k=a_n+1}^{b_n} \frac{1}{k} \quad \leq \quad \int_{a_n}^{b_n} \frac{dt}{t} \quad \leq \quad \sum_{k=a_n}^{b_n-1} \frac{1}{k},$$

which implies that

$$\frac{1}{a_n} \quad \geq \quad \left(\sum_{k=a_n}^{b_n} \frac{1}{k}\right) - \left(\int_{a_n}^{b_n} \frac{dt}{t}\right) \quad \geq \quad \frac{1}{b_n}.$$

The result follows by taking $n \to \infty$, since the hypotheses on a_n and b_n imply that $\lim_{n\to\infty} 1/a_n = 0$ and $\lim_{n\to\infty} 1/b_n = 0$. □

3. Proofs

3.1. Farey Staircase at 0

We are now ready to prove Theorem 1, which states that

$$\lim_{n\to\infty} B_n(x) = \sum_{k=1}^{\lfloor x \rfloor} \log(x/k), \tag{11}$$

if $x \geq 0$ and $B_n(x)$ denotes the function

$$B_n(x) \coloneqq \sum_{k=1}^{n} \tfrac{1}{k} \left\lfloor \tfrac{k}{n} x \right\rfloor = \left\lfloor \tfrac{1}{n} x \right\rfloor + \tfrac{1}{2} \left\lfloor \tfrac{2}{n} x \right\rfloor + \tfrac{1}{3} \left\lfloor \tfrac{3}{n} x \right\rfloor + \cdots + \tfrac{1}{n} \lfloor x \rfloor .$$

Proof of Theorem 1. For x in the range $0 \leq x < 1$, each summand in $B_n(x)$ vanishes so $B_n(x) = 0$.

Suppose we fix x in the range $1 \leq x < 2$. For $k \in \{1, 2, \ldots, n\}$, we have

$$\left\lfloor \tfrac{k}{n} x \right\rfloor = \begin{cases} 0 & \text{if } 1 \leq k < \frac{n}{x}, \\ 1 & \text{if } \frac{n}{x} \leq k \leq n. \end{cases}$$

Then

$$B_n(x) = \sum_{k=1}^{n} \tfrac{1}{k} \left\lfloor \tfrac{k}{n} x \right\rfloor = \sum_{k=\lceil n/x \rceil}^{n} \tfrac{1}{k}.$$

By Lemma 9,

$$\lim_{n\to\infty} \sum_{k=\lceil n/x \rceil}^{n} \frac{1}{k} = \lim_{n\to\infty} \int_{\lceil n/x \rceil}^{n} \frac{dt}{t} = \lim_{n\to\infty} \left(\int_{n/x}^{n} \frac{dt}{t} - \int_{n/x}^{\lceil n/x \rceil} \frac{dt}{t} \right) = \log x$$

since the integral $\int_{n/x}^{\lceil n/x \rceil} \frac{dt}{t} \leq \frac{x}{n}$ vanishes in the limit.

In general, suppose x is in the range $M \leq x < M + 1$ for a positive integer M. Then if we group the summands in $B_n(x) = \sum_{k=1}^{n} \frac{1}{k} \left\lfloor \frac{k}{n} x \right\rfloor$ according to the value of $\left\lfloor \frac{k}{n} x \right\rfloor$, we obtain

$$B_n(x) = \sum_{k=\lceil n/x \rceil}^{\lceil 2n/x \rceil - 1} \tfrac{1}{k} + \sum_{k=\lceil 2n/x \rceil}^{\lceil 3n/x \rceil - 1} \tfrac{2}{k} + \cdots + \sum_{k=\lceil (M-1)n/x \rceil}^{\lceil Mn/x \rceil - 1} \tfrac{M-1}{k} + \sum_{k=\lceil Mn/x \rceil}^{n} \tfrac{M}{k} \tag{12}$$

$$= \sum_{k=\lceil n/x \rceil}^{n} \tfrac{1}{k} + \sum_{k=\lceil 2n/x \rceil}^{n} \tfrac{1}{k} + \cdots + \sum_{k=\lceil Mn/x \rceil}^{n} \tfrac{1}{k}. \tag{13}$$

Then we apply Lemma 9 to each sum,

$$\begin{aligned}\lim_{n\to\infty} B_n(x) &= \sum_{j=1}^{M} \lim_{n\to\infty} \sum_{k=\lceil jn/x\rceil}^{n} \frac{1}{k} = \sum_{j=1}^{M} \lim_{n\to\infty} \int_{\lceil jn/x\rceil}^{n} \frac{dt}{t} \\ &= \sum_{j=1}^{M} \lim_{n\to\infty} \left(\int_{jn/x}^{n} \frac{dt}{t} - \int_{jn/x}^{\lceil jn/x\rceil} \frac{dt}{t} \right) = \sum_{j=1}^{M} \log(x/j).\end{aligned}$$

This completes the proof, since M was chosen to be $M = \lfloor x \rfloor$. □

3.2. Farey Staircase at p/q

We next address the fractal behavior of $A_n(x)$. For a reduced fraction p/q, what does the Farey staircase look like in a small neighborhood around p/q? The function

$$A_n\left(x + \frac{p}{q}\right) - A_n\left(\frac{p}{q}\right)$$

shifts the Farey staircase so that the point above p/q is moved to the origin. After zooming in by a factor of n, we will show these functions converge to a limit as $n \to \infty$.

Theorem 3 states that for any reduced fraction p/q and $x \geq 0$,

$$\lim_{n\to\infty} \left(B_n\left(x + \frac{p}{q}n\right) - B_n\left(\frac{p}{q}n\right) \right) = \frac{1}{q} \sum_{k=1}^{\lfloor qx \rfloor} \log\left(\frac{qx}{k}\right).$$

Recall from the introduction that $A_n(x) = \dfrac{1}{n} \displaystyle\sum_{k=1}^{n} \tfrac{1}{k} \lfloor kx \rfloor$ and $B_n(x) = \displaystyle\sum_{k=1}^{n} \tfrac{1}{k} \left\lfloor \tfrac{k}{n} x \right\rfloor$.

Proof of Theorem 3. The summand in the expression $B_n(x + \frac{p}{q}n) - B_n(\frac{p}{q}n)$ is $\frac{1}{k}\left\lfloor \frac{k}{n}x + \frac{kp}{q} \right\rfloor - \frac{1}{k}\left\lfloor \frac{kp}{q} \right\rfloor$. If $r \equiv kp \mod q$, then

$$\left\lfloor \frac{k}{n}x + \frac{kp}{q} \right\rfloor - \left\lfloor \frac{kp}{q} \right\rfloor = \left\lfloor \frac{k}{n}x + \frac{r}{q} \right\rfloor - \left\lfloor \frac{r}{q} \right\rfloor.$$

Moreover, if the integer r is chosen in the range $\{0, 1, 2, \ldots, q-1\}$, then $\left\lfloor \frac{r}{q} \right\rfloor = 0$. By grouping terms by the q-residue class of the product kp, we obtain

$$\begin{aligned}B_n(x + \tfrac{p}{q}n) - B_n(\tfrac{p}{q}n) &= \sum_{k=1}^{n} \frac{1}{k} \left(\left\lfloor \frac{k}{n}x + \frac{kp}{q} \right\rfloor - \left\lfloor \frac{kp}{q} \right\rfloor \right) && (14) \\ &= \sum_{r=0}^{q-1} \sum_{\substack{k=1 \\ kp\equiv r\,(q)}}^{n} \frac{1}{k} \left\lfloor \frac{k}{n}x + \frac{r}{q} \right\rfloor. && (15)\end{aligned}$$

For the inner summation, we have

$$\sum_{\substack{k=1\\ kp\equiv r\ (q)}}^{n} \frac{1}{k}\left\lfloor \frac{k}{n}x+\frac{r}{q}\right\rfloor = \sum_{\substack{k=1\\ kp\equiv r\ (q)}}^{n} \frac{1}{k}\sum_{j=1}^{\infty}\mathbb{1}\left(j\le \left\lfloor \frac{k}{n}x+\frac{r}{q}\right\rfloor\right) \tag{16}$$

$$= \sum_{j=1}^{\infty}\sum_{\substack{k=1\\ kp\equiv r\ (q)}}^{n} \frac{1}{k}\mathbb{1}\left(j\le \left\lfloor \frac{k}{n}x+\frac{r}{q}\right\rfloor\right) \tag{17}$$

where $\mathbb{1}$ denotes the indicator function. Since the index k ranges from 1 to n, the maximal value of j satisfying $j \le \left\lfloor \frac{k}{n}x+\frac{r}{q}\right\rfloor$ is equal to $j_{\max} = \left\lfloor x+\frac{r}{q}\right\rfloor$.

The condition $j \le \left\lfloor \frac{k}{n}x+\frac{r}{q}\right\rfloor$ can be solved for k as follows:

$$j\le \left\lfloor \frac{k}{n}x+\frac{r}{q}\right\rfloor \quad \text{if and only if} \quad j\le \frac{k}{n}x+\frac{r}{q} \quad \text{if and only if} \quad \frac{n}{x}\left(j-\frac{r}{q}\right)\le k.$$

Thus we can express the sum (17) as

$$\sum_{\substack{k=1\\ kp\equiv r\ (q)}}^{n} \frac{1}{k}\left\lfloor \frac{k}{n}x+\frac{r}{q}\right\rfloor = \sum_{j=1}^{j_{\max}}\sum_{\substack{k=1\\ kp\equiv r\ (q)}}^{n} \frac{1}{k}\mathbb{1}\left(k\ge \frac{n}{x}\left(j-\frac{r}{q}\right)\right) \tag{18}$$

$$= \sum_{j=1}^{\left\lfloor x+\frac{r}{q}\right\rfloor}\sum_{\substack{k=\lceil \frac{n}{x}(j-\frac{r}{q})\rceil\\ kp\equiv r\ (q)}}^{n} \frac{1}{k}. \tag{19}$$

Since the function $1/t$ is sufficiently smooth, the inner summation of (19) restricted to a single q-residue class converges to a $1/q$-factor of the unrestricted sum,

$$\lim_{n\to\infty}\sum_{\substack{k=\lceil \frac{n}{x}(j-\frac{r}{q})\rceil\\ kp\equiv r\ (q)}}^{n} \frac{1}{k} = \frac{1}{q}\left(\lim_{n\to\infty}\sum_{k=\lceil \frac{n}{x}(j-\frac{r}{q})\rceil}^{n} \frac{1}{k}\right) = \frac{1}{q}\log\left(\frac{qx}{qj-r}\right).$$

Summing these limits over the index j as in (19),

$$\lim_{n\to\infty}\sum_{\substack{k=1\\ kp\equiv r\ (q)}}^{n} \frac{1}{k}\left\lfloor \frac{k}{n}x+\frac{r}{q}\right\rfloor = \sum_{j=1}^{\left\lfloor x+\frac{r}{q}\right\rfloor}\lim_{n\to\infty}\sum_{\substack{k=\frac{n}{x}(j-\frac{r}{q})\\ kp\equiv r\ (q)}}^{n} \frac{1}{k} = \frac{1}{q}\sum_{j=1}^{\left\lfloor x+\frac{r}{q}\right\rfloor}\log\left(\frac{qx}{qj-r}\right). \tag{20}$$

Finally, we sum over the q-residue class representatives r as in (15),

$$\lim_{n\to\infty}\left(B_n\left(x+\frac{p}{q}n\right)-B_n\left(\frac{p}{q}n\right)\right) = \sum_{r=0}^{q-1}\frac{1}{q}\sum_{j=1}^{\lfloor x+\frac{r}{q}\rfloor}\log\left(\frac{qx}{qj-r}\right) \tag{21}$$

$$= \frac{1}{q}\sum_{r=0}^{q-1}\sum_{\substack{\ell=1\\ \ell\equiv -r(q)}}^{\lfloor qx\rfloor}\log\left(\frac{qx}{\ell}\right) \tag{22}$$

$$= \frac{1}{q}\sum_{\ell=1}^{\lfloor qx\rfloor}\log\left(\frac{qx}{\ell}\right) \tag{23}$$

as claimed. □

3.3. Incremented Staircase at 0

Concerning the limiting behavior of $D_n(x) = A_n(x) - A_{n-1}(x)$, Theorem 4 states that

$$\lim_{n\to\infty} n^2 D_n\left(\frac{1}{n}x\right) = \sum_{k=1}^{\lfloor x\rfloor}\left(1-\log\left(\frac{x}{k}\right)\right).$$

To address this limit, we combine Theorem 1 with Proposition 7.

Proof of Theorem 4. From Proposition 7, we have

$$(n^2-n)D_n\left(\frac{1}{n}x\right) = \lfloor x\rfloor - nA_n\left(\frac{1}{n}x\right) = \lfloor x\rfloor - B_n(x).$$

In the limit $n\to\infty$, the right-hand side is equal to

$$\lfloor x\rfloor - B(x) = \sum_{k=1}^{\lfloor x\rfloor}\left(1-\log\left(\frac{x}{k}\right)\right).$$

To complete the proof, it suffices to verify the uniform convergence $nD_n(x)\to 0$ as $n\to\infty$. This follows from Lemma 8 (b), which states

$$0 \le nD_n(x) \le \frac{\log n}{n-1} \qquad \text{for all } n\ge 2.$$ □

4. The Dilation Derivative

In this section, we introduce some additional conceptual framework that suggests a heuristic argument for Theorem 1. We do not turn this heuristic into a rigorous alternative proof.

Definition 10. Given a function $f : \mathbb{R} \to \mathbb{R}$, let

$$\Delta f(x) = \lim_{\lambda \to 1} \frac{\lambda^{-1} f(\lambda x) - f(x)}{\lambda - 1}, \tag{24}$$

when the limit exists. We call Δf the *dilation derivative* of f.

Compared to the limit definition of the usual derivative, the horizontal translation $f(x + \epsilon)$ is replaced with the dilation $\lambda^{-1} f(\lambda x)$, with $\lambda = 1 + \epsilon$.

Example 11. The dilation derivative satisfies the following identities.

1. For any function f, we have $\Delta f(0) = -f(0)$;
2. $\Delta x^\alpha = (\alpha - 1)x^\alpha$;
3. $\Delta e^{kx} = (kx - 1)e^{kx}$;
4. $\Delta \log x = 1 - \log x$.

These examples generalize to the following result.

Proposition 12. *For a differentiable function $f(x)$,*

$$\Delta f(x) = x \frac{d}{dx} f(x) - f(x).$$

Proof. If $x = 0$, the claim follows from the definition of $\Delta f(x)$. Now suppose $x \neq 0$. We have

$$\begin{aligned} \Delta f(x) = \lim_{\lambda \to 1} \frac{1}{\lambda} \cdot \frac{f(\lambda x) - \lambda f(x)}{\lambda - 1} &= \lim_{\lambda \to 1} \frac{f(\lambda x) - f(x) + f(x) - \lambda f(x)}{\lambda - 1} && (25) \\ &= \left(\lim_{\lambda \to 1} \frac{f(\lambda x) - f(x)}{\lambda - 1} \right) - f(x) && (26) \\ &= x \left(\lim_{\lambda \to 1} \frac{f(x + (\lambda - 1)x) - f(x)}{(\lambda - 1)x} \right) - f(x). && (27) \end{aligned}$$

The limit in the last expression is $\frac{d}{dx} f(x)$, since $\epsilon := (\lambda - 1)x \to 0$ as $\lambda \to 1$. □

Claim 13 (Dilation Heuristic)**.** *Suppose that $B(x)$ is the pointwise limit of $B_n(x)$. Then*

$$\Delta B(x) = \lfloor x \rfloor - B(x).$$

Proposition 7 implies that

$$(n - 1)A_n(x) - (n - 1)A_{n-1}(x) = \frac{1}{n} \left(\lfloor nx \rfloor - nA_n(x) \right).$$

Replace x with $\frac{1}{n-1}x$, to yield

$$\frac{n-1}{n}B_n\left(\frac{n}{n-1}x\right)-B_{n-1}(x)=\frac{1}{n}\left(\left\lfloor\frac{n}{n-1}x\right\rfloor-B_n\left(\frac{n}{n-1}x\right)\right).$$

Now let $\lambda_n=\frac{n}{n-1}$, so the equation above multiplied by $n-1$ becomes

$$\frac{\lambda_n^{-1}B_n(\lambda_n x)-B_{n-1}(x)}{\lambda_n-1}=\lambda_n^{-1}\left(\lfloor\lambda_n x\rfloor-B_n(\lambda_n x)\right). \tag{28}$$

Then take the limit as $n\to\infty$, so that $\lambda_n\to 1$. The right-hand side approaches $\lfloor x\rfloor-B(x)$. The left-hand side is more subtle: by hypothesis $B_n\to B$, so we may compare the left-hand side with

$$\lim_{\lambda_n\to 1}\frac{\lambda_n^{-1}B(\lambda_n x)-B(x)}{\lambda_n-1}=\Delta B(x). \tag{29}$$

We would like to say that as $n\to\infty$, the left-hand side of (28) approaches $\Delta B(x)$. However, the key step that remains in order to reach this conclusion rigorously is to bound the difference

$$\left|\frac{\lambda_n^{-1}B_n(\lambda_n x)-B_{n-1}(x)}{\lambda_n-1}-\frac{\lambda_n^{-1}B(\lambda_n x)-B(x)}{\lambda_n-1}\right|.$$

To do so, it is necessary to further analyze the rate of convergence $B_n(x)\to B(x)$, which we leave for future investigation.

Corollary 14 (to Claim 13)**.** *The limit function $B(x)$ is a solution to the differential equation*

$$xB'(x)=\lfloor x\rfloor, \qquad B(0)=0.$$

If we know additionally that the limit function $B(x)$ is continuous, this implies that $B(x)=\displaystyle\int_0^x\frac{\lfloor t\rfloor}{t}\,dt$, recovering Theorem 1 via Remark 2 (ii).

A. Proof of Olympiad Problem

We conclude with a proof of the Olympiad problem (9) for the sake of completeness, which we restate below.

Claim 15 (1981 USAMO, Problem 5)**.** *For any positive integer n and any x,*

$$\lfloor nx\rfloor\geq\sum_{k=1}^{n}\frac{1}{k}\lfloor kx\rfloor.$$

Proof. We proceed by induction on n. The relation clearly holds when $n = 1$ for all x, so suppose $n \geq 2$.

Given n and x, let $d = d(n,x)$ denote any element of $\{1, 2, \ldots, n\}$ such that

$$\tfrac{1}{d} \lfloor dx \rfloor = \max \left\{ \tfrac{1}{k} \lfloor kx \rfloor : k = 1, 2, \ldots, n \right\}.$$

Note that $\lfloor x + y \rfloor \geq \lfloor x \rfloor + \lfloor y \rfloor$ for any real numbers x and y, so we have

$$\lfloor nx \rfloor \geq \lfloor (n-d)x \rfloor + \lfloor dx \rfloor, \tag{30}$$

and we can bound each of the right-hand terms separately.

The assumption that $\frac{1}{d} \lfloor dx \rfloor \geq \frac{1}{k} \lfloor kx \rfloor$ for $k = 1, 2, \ldots, n$ implies that

$$\lfloor dx \rfloor = \sum_{k=n-d+1}^{n} \frac{1}{d} \lfloor dx \rfloor \geq \sum_{k=n-d+1}^{n} \frac{1}{k} \lfloor kx \rfloor, \tag{31}$$

and by the induction hypothesis,

$$\lfloor (n-d)x \rfloor \geq \sum_{k=1}^{n-d} \frac{1}{k} \lfloor kx \rfloor. \tag{32}$$

Combining (30) with the bounds (31) and (32) implies that

$$\lfloor nx \rfloor \geq \left(\sum_{k=1}^{n-d} \tfrac{1}{k} \lfloor kx \rfloor \right) + \left(\sum_{k=n-d+1}^{n} \tfrac{1}{k} \lfloor kx \rfloor \right)$$

as desired. □

Acknowledgements. I am grateful to Michael Filaseta for bringing this problem to my attention, and to Jeffrey Lagarias and Samantha Pinella for useful conversations. I thank the anonymous referee and editors for suggestions that improved this work, in particular in the proof of Theorem 4. The figures in this paper were made using the Matplotlib [3] Python library. This work was supported by NSF grants DMS-1600223 and DMS-1701576 and a Rackham Predoctoral Fellowship.

References

[1] S. Aursukaree, T. Khemaratchatakumthorn, and P. Pongsriiam, Generalizations of Hermite's identity and applications, *Fibonacci Q.*, **57** (2) (2019), 126–133.

[2] F. Dress, Discrepancy of Farey sequences, *J. Théor. Nombres Bordx.*, **11** (2) (1999), 345–367.

[3] J. D. Hunter, Matplotlib: A 2D graphics environment, *Computing in Science & Engineering*, **9** (3) (2007), 90–95.

[4] S. Kanemitsu and M. Yoshimoto, Farey series and the Riemann hypothesis, *Acta Arith.*, **75** (4) (1996), 351–374.

[5] M. S. Klamkin, *U.S.A. Mathematical Olympiads, 1972–1986*, volume 33 of *New Mathematical Library*, Mathematical Association of America, Washington, DC, 1988.

[6] M. Kunik, A scaling property of Farey fractions, *Eur. J. Math.*, **2** (2) (2016), 383–417.

[7] J. C. Lagarias and H. Mehta, Products of Farey fractions, *Exp. Math.*, **26** (1) (2017), 1–21.

[8] L. C. Larson, Solutions to 1981 U.S.A. and Canadian Mathematical Olympiads, *Math. Mag.*, **54** (5) (1981), 277–280.

[9] H. Niederreiter, The distribution of Farey points, *Math. Ann.*, **201** (1973), 341–345.

[10] S. Ramanujan, Question 723, *J. Indian Math. Soc.*, **10** (1918), 357–358.

[11] D. R. Richman, A sum involving the greatest-integer function, *Integers*, **19** (2019), #A42.

[12] S. T. Somu and A. Kukla, On some generalizations to floor function identities of Ramanujan, *Integers*, **22** (2022), #A33.

[13] T. Thanatipanonda and E. Wong, Curious bounds for floor function sums, *J. Integer Seq.*, **21** (1) (2018), #18.1.8.