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ON UNIT-WEIGHTED ZERO-SUM CONSTANTS OF $\mathbb{Z}_n$

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Abstract

Given a subset $A \subseteq \mathbb{Z}_n$, the constant C_A is defined to be the smallest natural number k such that any sequence of k elements in $\mathbb{Z}_n$ has an A-weighted zero-sum subsequence having consecutive terms. The value of $C_{U(n)}$ is known for every odd number n. We determine the value of $C_{U(n)}$ for every even number n. We define a C-extremal sequence for $U(n)$ to be a sequence in $\mathbb{Z}_n$ having length $C_{U(n)} - 1$ which does not have any $U(n)$-weighted zero-sum subsequence having consecutive terms. We characterize the C-extremal sequences for $U(n)$ when n is a power of 2. We also determine the value of C_A when A is the set of all odd (or all even) elements of $\mathbb{Z}_n$, and when $A = \{1, 2, \ldots, r\}$ for some $r \leq n - 1$.

1. Introduction

We denote the number of elements in a finite set S by $|S|$. For $a, b \in \mathbb{Z}$, we denote the set $\{k \in \mathbb{Z} : a \leq k \leq b\}$ by $[a, b]$. In this section, R will denote a ring with unity. The next three definitions are given in [9].

Definition 1.1. Let M be an R-module and $A \subseteq R$. Let $S = (x_1, x_2, \ldots, x_k)$ be a sequence in M. A subsequence T is called an *A-weighted zero-sum subsequence* if

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the set $J = \{i : x_i \in T\}$ is non-empty, and for each $i \in J$ there exist $a_i \in A$ such that $\sum_{i\in J} a_i x_i = 0$.

Definition 1.2. Let M be a finite R-module and $A \subseteq R$. The *A-weighted Davenport constant* D_A is the least positive integer k such that every sequence in M of length k has an A-weighted zero-sum subsequence.

Definition 1.3. Let M be a finite R-module and $A \subseteq R$. The *A-weighted constant* C_A is the least positive integer k such that every sequence S in M of length k has an A-weighted zero-sum subsequence whose terms are consecutive terms of S.

Definition 1.4. Let M be a finite R-module and $A \subseteq R$. The *A-weighted Gao constant* E_A is the least positive integer k such that every sequence in M of length k has an A-weighted zero-sum subsequence of length $|M|$.

Remark 1.5. We denote the above three constants by $D(M), C(M)$, and $E(M)$, respectively, when $A = \{1\}$. In [9] we show that for every finite R-module M we have $D_A \le C_A \le |M|$. Also, it is easy to see that $E_A \ge D_A + |M| - 1$.

We denote the ring $\mathbb{Z}/n\mathbb{Z}$ by $\mathbb{Z}_n$. When $A \subseteq \mathbb{Z}_n$, Yuan and Zeng show in [16] that $E_A = D_A + n - 1$. Some results for these constants when $M = R = \mathbb{Z}_n$ are given in [1], [3], [6], [8], [9], and [15].

Definition 1.6. Let $A \subseteq \mathbb{Z}_n$. A sequence in $\mathbb{Z}_n$ of length $E_A - 1$ which does not have any A-weighted zero-sum subsequence of length n is called an *E-extremal sequence for A*. A sequence in $\mathbb{Z}_n$ of length $C_A - 1$ which does not have any A-weighted zero-sum subsequence of consecutive terms is called a *C-extremal sequence for A*.

Let $U(n)$ denote the group of units in $\mathbb{Z}_n$ and $U(n)^k = \{x^k : x \in U(n)\}$. We let $\Omega(n) = r_1 + \cdots + r_s$ if $n = p_1^{r_1} \dots p_s^{r_s}$ where the p_i's are distinct primes.

For a finite abelian group G we show that $C(G) = |G|$. This result is in contrast with the value of $D(G)$ for a finite abelian group G, as in Corollary 1.1 of [14] we see that $D(G) = |G|$ if and only if G is a finite cyclic group. As a consequence of the fact that $C(\mathbb{Z}_2^a) = 2^a$, in Theorem 3.6 we show that for every positive integer n we have that $C_{U(n)} = 2^{\Omega(n)}$. Thus, we obtain a new proof of Corollary 4 of [9] while also generalizing it.

The result that $D(\mathbb{Z}_2^a) = a + 1$ for every positive integer a is a special case of the main result of Olson in [13]. As a consequence of this we rederive the result that $D_{U(n)} = \Omega(n) + 1$ for every positive integer n (without using the value of $E_{U(n)}$). This can also be arrived at by using the result of Yuan and Zeng in [16], as the values of $E_{U(n)}$ for every positive integer n are given in [6] and [8].

However, we feel that it is of interest to derive the value of $D_{U(n)}$ for every positive integer n without using the value of $E_{U(n)}$, since for other weight-sets $A \subseteq \mathbb{Z}_n$, the values of D_A have been determined without using the corresponding values of E_A.

2. Some General Results

Theorem 2.1. *Let M be a finite R-module and N be a finite R'-module. Suppose $A \subseteq R$ and $B \subseteq R'$. Then we have that $D_{A\times B} \geq D_A + D_B - 1$ where we consider $M \times N$ as a module over $R \times R'$.*

Proof. Let $k = D_A - 1$ and $l = D_B - 1$. Suppose $S_1 = (x_1, \ldots, x_k)$ is a sequence in M which does not have any A-weighted zero-sum subsequence, and $S_2 = (y_1, \ldots, y_l)$ is a sequence in N which does not have any B-weighted zero-sum subsequence. Let S be the sequence in $M \times N$ defined as

$$\big(\,(x_1, 0), (x_2, 0), \ldots, (x_{k-1}, 0), (x_k, 0), (0, y_1), (0, y_2), \ldots, (0, y_{l-1}), (0, y_l)\,\big).$$

Suppose S has an $A \times B$-weighted zero-sum subsequence T. If no term of T is of the form $(0, y_j)$ for any $j \in [1, l]$, then we get the contradiction that S_1 has an A-weighted zero-sum subsequence. If T has a term of the form $(0, y_j)$ for some $j \in [1, l]$, then by taking the projection to the second coordinate, we get the contradiction that S_2 has a B-weighted zero-sum subsequence. Thus, we see that S does not have any $A \times B$-weighted zero-sum subsequence. Hence, we get that $D_{A\times B} \geq k + l + 1$ and so $D_{A\times B} \geq D_A + D_B - 1$. □

By a similar argument we get the next result. Here we mention the modules in the notation for the constant D_A as the set $A \subseteq R$ is used as a weight-set for the R-modules M, N, and $M \times N$.

Theorem 2.2. *Let M and N be R-modules and $A \subseteq R$. Then $M \times N$ is an R-module and we have $D_A(M \times N) \geq D_A(M) + D_A(N) - 1$.*

Theorem 2.3. *Let M be a finite R-module and N be a finite R'-module. Suppose $A \subseteq R$ and $B \subseteq R'$. Then we have that $C_{A\times B} \geq C_A\, C_B$ where we consider $M \times N$ as a module over $R \times R'$.*

Proof. Let $k = C_A - 1$ and $l = C_B - 1$. Suppose $S_1 = (x_1, \ldots, x_k)$ is a sequence in M which does not have any A-weighted zero-sum subsequence of consecutive terms and $S_2 = (y_1, \ldots, y_l)$ is a sequence in N which does not have any B-weighted zero-sum subsequence of consecutive terms. Consider the sequence S in $M \times N$ defined as

$$\big(S_1', (0, y_1), S_1', (0, y_2), S_1', \ldots, (0, y_l), S_1'\big) \text{ where } S_1' \text{ denotes } (x_1, 0), \ldots, (x_k, 0).$$

Suppose S has an $A\times B$-weighted zero-sum subsequence T having consecutive terms. If no term of T is of the form $(0, y_j)$ for any $j \in [1, l]$, then we get the contradiction that S_1 has an A-weighted zero-sum subsequence of consecutive terms. If T has a term of the form $(0, y_j)$ for some $j \in [1, l]$, then by taking the projection to the second coordinate, we get the contradiction that S_2 has a B-weighted zero-sum subsequence of consecutive terms. Thus, we see that S does not have any $A \times B$-weighted zero-sum subsequence of consecutive terms. Hence, it follows that $C_{A\times B} \geq k(l+1) + l + 1$ and so $C_{A\times B} \geq C_A\, C_B$. □

By a similar argument we get the next result. Here we mention the modules in the notation for the constant C_A as the set $A \subseteq R$ is used as a weight-set for the R-modules M, N, and $M \times N$.

Theorem 2.4. *Let M and N be R-modules and $A \subseteq R$. Then $M \times N$ is an R-module, and we have $C_A(M \times N) \geq C_A(M)\, C_A(N)$.*

3. The Determination of $C_{U(n)}$

For a positive integer n and for a prime p, we use the notation $v_p(n) = r$ to mean that $p^r \mid n$ and $p^{r+1} \nmid n$.

Theorem 3.1. *For any finite abelian group G we have that $C(G) = |G|$.*

Proof. By induction on the rank of G, Corollary 1 of [9], and Theorem 2.4, we can show that $C(G) \geq |G|$. Also, from Theorem 1 of [9] we get that $C(G) \leq |G|$. □

Let $x = (x_1, \ldots, x_m), y = (y_1, \ldots, y_m) \in \mathbb{Z}_2^m$. We define the dot product of x and y to be $x \cdot y = x_1y_1 + \cdots + x_my_m$.

Lemma 3.2. *Let a, m be positive integers such that $m \geq 2^a$ and let $v_1, \ldots, v_a \in \mathbb{Z}_2^m$. Then there exists a non-zero vector $w \in \mathbb{Z}_2^m$ such that for each $i \in [1, a]$ we have $w \cdot v_i = 0$ and the coordinates of w which are equal to one occur in consecutive positions.*

Proof. Let P be the matrix of size $m \times a$ whose columns are the vectors v_i. If $w \in \mathbb{Z}_2^m$, then wP is the vector in $\mathbb{Z}_2^a$ which is the sum of those rows of the matrix P which correspond to the coordinates of w which are 1. From Theorem 3.1 we see that $C(\mathbb{Z}_2^a) = 2^a$. As $m \geq 2^a$, the sum of some consecutive rows of P is zero. Thus, we can find a vector $w \in \mathbb{Z}_2^m$ as in the statement of the lemma. □

Lemma 3.3. *Let a and m be positive integers such that $m \geq 2^a$ and let $X_1, \ldots, X_a$ be subsets of $[1, m]$. Then there exists $Y \subseteq \{1, \ldots, m\}$ such that the elements of Y are consecutive numbers, and for each $i \in [1, a]$ we have that $|Y \cap X_i|$ is even.*

Proof. We identify a subset $A \subseteq [1, m]$ with the vector $x_A \in \mathbb{Z}_2^m$ whose j^{th} coordinate is one if and only if $j \in A$. For any two subsets A and B of $[1, m]$ we observe that $x_A \cdot x_B = 0$ if and only if $|A \cap B|$ is even. So we see that Lemma 3.3 follows from Lemma 3.2. □

For a divisor m of n, we define the natural map $f_{n,m} : \mathbb{Z}_n \to \mathbb{Z}_m$ to be the map given by $f_{n,m}\big(x + n\mathbb{Z}\big) = x + m\mathbb{Z}$. Let p be a prime divisor of n with $v_p(n) = r$. For a sequence S in $\mathbb{Z}_n$, we denote its image under f_{n,p^r} by $S^{(p)}$.

Observation 3.4 ([6])**.** Let n be a positive integer and let S be a sequence in $\mathbb{Z}_n$. Then S is a $U(n)$-weighted zero-sum sequence if and only if for every prime divisor p of n the sequence $S^{(p)}$ is a $U(p^r)$-weighted zero-sum sequence in $\mathbb{Z}_{p^r}$ where $r = v_p(n)$.

This is Observation 2.2 in [6]. The next result follows from Lemma 2.4 in [6] along with the remark which follows it.

Lemma 3.5 ([6])**.** *Let p be a prime and let $n = p^r$ for some r. Suppose $S = (x_1, \ldots, x_m)$ is a sequence in $\mathbb{Z}_n$ such that for each $i \in [1, r]$ the set X_i has an even number of elements where $X_i = \{\, j : x_j \not\equiv 0 \pmod{p^i} \,\}$. Then S is a $U(n)$-weighted zero-sum sequence.*

Theorem 3.6. *For every positive integer n we have that $C_{U(n)} = 2^{\Omega(n)}$.*

Proof. By Corollary 2 of [9] we see that $C_{U(n)} \geq 2^{\Omega(n)}$ for every positive integer n. Let $S = (x_1, \ldots, x_m)$ be a sequence in $\mathbb{Z}_n$ of length $m = 2^a$ where $a = \Omega(n)$. If we show that S has a $U(n)$-weighted zero-sum subsequence of consecutive terms, it will follow that $C_{U(n)} \leq 2^{\Omega(n)}$. For any prime divisor p of n and for each $i \in [1, v_p(n)]$, we let

$$X_i^{(p)} = \{\, j : x_j \not\equiv 0 \pmod{p^i} \,\}.$$

We observe that we have a sets in the collection

$$\big\{\, X_i^{(p)} : p \text{ is a prime divisor of } n \text{ and } i \in [1, v_p(n)] \,\big\}.$$

By Lemma 3.3 we have a subset $Y \subseteq [1, m]$ such that all the elements of Y are consecutive numbers, and for every prime divisor p of n and for every $i \in [1, v_p(n)]$ we have $|\, Y \cap X_i^{(p)} \,|$ is even. Let T be the subsequence of S such that x_j is a term of T if and only if $j \in Y$.

Let q be a prime divisor of n and $v_q(n) = r$. By Lemma 3.5 the sequence $T^{(q)}$ is a $U(q^r)$-weighted zero-sum sequence in $\mathbb{Z}_{q^r}$, as for each $i \in [1, r]$ the set

$$Y \cap X_i^{(q)} = \{\, j \in Y : x_j \not\equiv 0 \pmod{q^i} \,\}$$

has even size. Thus, by Observation 3.4 it follows that the sequence T is a $U(n)$-weighted zero-sum sequence. Hence, we see that S has a $U(n)$-weighted zero-sum subsequence of consecutive terms. $\square$

4. The Determination of $D_{U(n)}$

The next result is a special case of the main result in [13] and follows easily by considering $\mathbb{Z}_2^a$ as a $\mathbb{Z}_2$-vector space.

Theorem 4.1. *For every positive integer a we have that $D(\mathbb{Z}_2^a) = a + 1$.*

We can also prove Theorem 4.1 by using Theorem 2.2 and the observation that $D(\mathbb{Z}_2) = 2$. Related results can be found in Chapter 5 of [5].

Lemma 4.2. *Let a, m be positive integers such that $m \geq a+1$ and $v_1, \ldots, v_a \in \mathbb{Z}_2^m$. Then there exists a non-zero vector $w \in \mathbb{Z}_2^m$ such that for every $i \in [1, a]$ we have that $w \cdot v_i = 0$.*

Proof. Let P be the matrix of size $m \times a$ whose columns are the vectors v_i. If $w \in \mathbb{Z}_2^m$, then wP is the vector in $\mathbb{Z}_2^a$ which is the sum of those rows of the matrix P which correspond to the coordinates of w which are 1. From Theorem 4.1 we have $D(\mathbb{Z}_2^a) = a + 1$. As $m \geq a + 1$, the sum of some rows of P is zero. Thus, we can find a vector $w \in \mathbb{Z}_2^m$ as in the statement of the lemma. $\square$

The proof of the next result is similar to the proof of Lemma 3.3.

Lemma 4.3. *Let a, m be positive integers such that $m \geq a + 1$ and let $X_1, \ldots, X_a$ be subsets of $[1, m]$. Then there exists a non-empty subset $Y \subseteq [1, m]$ such that for every $i \in [1, a]$ we have that $|\, Y \cap X_i \,|$ is even.*

Theorem 4.4. *For every positive integer n we have that $D_{U(n)} = \Omega(n) + 1$.*

Proof. For every prime p, the argument in the proof of Theorem 2 of [9] shows that $D_{U(p)} = 2$. So from Lemma 1.8 of [11] we see that $D_{U(n)} \geq \Omega(n) + 1$ for every positive integer n.

Let $S = (x_1, \ldots, x_m)$ be a sequence in $\mathbb{Z}_n$ of length $m = \Omega(n)+1$. If we show that S has a $U(n)$-weighted zero-sum subsequence, it will follow that $D_{U(n)} \leq \Omega(n) + 1$. For every prime divisor p of n and for every $i \in [1, v_p(n)]$, let

$$X_i^{(p)} = \{\, j : x_j \not\equiv 0 \pmod{p^i} \,\}.$$

The collection

$$\{\, X_i^{(p)} : p \text{ is a prime divisor of } n \text{ and } i \in [1, v_p(n)] \,\}$$

has $\Omega(n)$ sets. As $m = \Omega(n) + 1$, by Lemma 4.3 there exists a non-empty subset $Y \subseteq [1, m]$ such that for each prime divisor p of n and for each $i \in [1, v_p(n)]$, the set $Y \cap X_i^{(p)}$ has an even number of elements. Let T be the subsequence of S such that x_j is a term of T if and only if $j \in Y$.

Let p be a prime divisor of n and let $v_p(n) = r$. By Lemma 3.5 we see that the sequence $T^{(p)}$ is a $U(p^r)$-weighted zero-sum sequence in $\mathbb{Z}_{p^r}$, since for each $i \in [1, r]$ the set

$$Y \cap X_i^{(p)} = \{\, j \in Y : x_j \not\equiv 0 \pmod{p^i} \,\}$$

has even size. Thus, by Observation 3.4 the sequence T is a $U(n)$-weighted zero-sum sequence. Hence, it follows that S has a $U(n)$-weighted zero-sum subsequence. $\square$

5. Zero-Sum Subsequences of Length m in $\mathbb{Z}_2^a$

If T is a subsequence of a sequence S, then $S - T$ denotes the sequence which is obtained by removing the terms of T from S.

Observation 5.1. Let $(G, +)$ be a finite abelian group of order n and $x \in G$. Let S be a sequence of length n in G and $S - x$ denote the sequence which is obtained by subtracting x from each term of S. If $S - x$ is a zero-sum sequence, then S is a zero-sum sequence.

The next result is equivalent to Lemma 2.6 of [6]. This can be shown by a similar reasoning as in the remark which immediately follows Corollary 3.2 of [6].

Theorem 5.2. *Let a, m be positive integers such that $m \geq 2^a$ and m is even. Suppose S is a sequence in $\mathbb{Z}_2^a$ of length $m + a$. Then S has a zero-sum subsequence of length m.*

Proof. Let $m \geq 2^a$ and $S = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_2^a$ of length $k = m + a$. Suppose each term of S occurs an even number of times in S. As $x + x = 0$ for any $x \in \mathbb{Z}_2^a$, it follows that S has a zero-sum subsequence of any even length, and hence also of length m. So we may assume that some term of S occurs an odd number of times. By Observation 5.1 we can assume that that term is 0.

Let S' be the unique subsequence of S whose terms are all distinct, and such that each term of $S - S'$ occurs an even number of times in $S - S'$. Let the length of S' be k'. Suppose $k' \leq a$. It follows that $k - k' = (m + a) - k'$ and so $m \leq k - k'$. As m is even, we see that $S - S'$ (and hence S) has a zero-sum subsequence of length m.

So we may assume that $k' \geq a + 1$. By Theorem 4.1 we see that $D(\mathbb{Z}_2^a) = a + 1$ and hence it follows that S' has a zero-sum subsequence. Let T be a zero-sum subsequence of S' having the largest length. As 0 occurs an odd number of times in S, it is a term of S'. So we see that 0 is a term of T. Let the length of T be l. As $D(\mathbb{Z}_2^a) = a + 1$ and $S' - T$ does not have any zero-sum subsequence, it follows that $k' - l \leq a$. We now claim that

$$l \leq m \leq (k - k') + l. \tag{1}$$

As all terms of S' are distinct, it follows that $k' \leq 2^a$. So as $l \leq k'$ and $2^a \leq m$, we see that $l \leq m$. As $k' - l \leq a$ and $m = k - a$, we have that $m \leq (k - k') + l$. This proves (1).

Suppose l is even. From (1) we see that $m - l \leq k - k'$. As $m - l$ is even and each term of $S - S'$ occurs an even number of times, it follows that $S - S'$ has a zero-sum subsequence having length $m - l$. As T is a zero-sum subsequence of S' having length l, we get a zero-sum subsequence of S having length m.

Suppose l is odd. As both m and $k - k'$ are even, from (1) it follows that $m \leq (k - k') + l - 1$. As $m - (l - 1)$ is even and each term of $S - S'$ occurs an even number of times, it follows that $S - S'$ has a zero-sum subsequence having length $m - (l - 1)$. As 0 is a term of T which has length l, we get a zero-sum subsequence of S' having length $l - 1$. So we get a zero-sum subsequence of S having length m. □

Lemma 5.3. *Let a, m be positive integers such that $m \geq 2^a$ and m is even. Suppose $v_1, \dots, v_a \in \mathbb{Z}_2^{m+a}$. Then there exists a non-zero vector $w \in \mathbb{Z}_2^{m+a}$ such that exactly m coordinates of w are one and for every $i \in [1, a]$ we have that $w \cdot v_i = 0$.*

Proof. The proof of this lemma is similar to the proof of Lemma 4.2 where we use Theorem 5.2 in place of Theorem 4.1. □

The next result follows easily from Lemma 5.3. It is Lemma 2.5 of [6].

Lemma 5.4 ([6])**.** *Let a, m be positive integers such that $m \geq 2^a$ and m is even. Suppose $X_1, \dots, X_a$ are subsets of $[1, m + a]$. Then there exists a non-empty subset Y of $[1, m + a]$ such that $|Y| = m$, and for every $i \in [1, a]$ we have that $|\, Y \cap X_i \,|$ is even.*

Theorem 5.5. *Let n be a positive integer and m be an even integer which is at least $2^{\Omega(n)}$. Suppose S is a sequence in $\mathbb{Z}_n$ having length $m + \Omega(n)$. Then S has a $U(n)$-weighted zero-sum subsequence of length m.*

Proof. The proof of this theorem is similar to the proof of Theorem 4.4, where we use Lemma 5.4 in place of Lemma 4.3. □

Remark 5.6. The proof of Theorem 1.3 (ii) of [6] also goes through when n is odd. In this sense, we see that Theorem 5.5 is the same as Theorem 1.3 (ii) of [6].

Corollary 5.7. *Let a be a positive integer and let m be an even integer which is at least 2^a. Then every m-dimensional subspace of $\mathbb{Z}_2^{m+a}$ has a vector v such that exactly m coordinates of v are equal to one.*

Proof. Let W be an m-dimensional subspace of $\mathbb{Z}_2^{m+a}$. Then the dimension of $W^\perp$ is a. Let $\{v_1, \dots, v_a\}$ be a basis of $W^\perp$ and A be the $a \times (m + a)$ matrix whose rows are the basis vectors of $W^\perp$. Then A gives a map $\mathbb{Z}_2^{m+a} \to \mathbb{Z}_2^a$ whose kernel is $(W^{\perp})^{\perp}$ which can be shown to be W. For each $j \in [1, m+a]$, let C_j denote the j^{th} column of A. By Theorem 5.2 we can find a subset $I \subseteq [1, m+a]$ such that $|I| = m$ and $\sum_{i \in I} C_i = 0$. Let $v = (x_1, \ldots, x_{m+a}) \in \mathbb{Z}_2^{m+a}$ be such that $x_i = 1$ if and only if $i \in I$. As we have that $Av = \sum_{i \in I} C_i$, it follows that $v \in ker\ A = W$. Hence, we see that W has a vector v such that exactly m coordinates of v are equal to one. □

Definition 5.8. For every $v \in \mathbb{Z}_2^n$ we define the *weight of* v to be the number of coordinates of v which are equal to one. Let $l(n, m)$ denote the *largest possible dimension* of a subspace of $\mathbb{Z}_2^n$ which does not have any vector of weight m, i.e.,

$$l(n, m) = \max \{\dim(W) : W \text{ a subspace of } \mathbb{Z}_2^n \text{ with } \mathrm{wt}(v) \neq m \text{ for every } v \in W\}.$$

Corollary 5.9. *Let a be a positive integer and let m be an even integer which is at least 2^a. Then we have that $l(n, m) = m - 1$ where $n = m + a$.*

Proof. Let $\{e_1, \ldots, e_n\}$ be the standard basis of $\mathbb{Z}_2^n$. The span of $\{e_1, \ldots, e_{m-1}\}$ does not have any vector of weight m, and hence it follows that $l(n, m) \geq m - 1$. By Corollary 5.7 we see that if m is an even integer which is at least 2^a, then every m dimensional subspace of $\mathbb{Z}_2^n$ has a vector of weight m. Hence, it follows that $l(n, m) \leq m - 1$. □

Corollary 5.9 is a weaker version of Theorem 1.1 of [4].

6. Extremal Sequences for $U(n)$ where $n = 2^k$

In [2], the D-extremal sequences for $U(n)$ were characterized when n is odd and when n is a power of 2. In [10], the C-extremal sequences for $U(n)$ have been characterized when n is odd. In [12], the E-extremal sequences for $U(n)$ have been characterized when n is odd and when $n = 2^r p$ where p is an odd prime. In this section, we characterize the C-extremal and E-extremal sequences for $U(n)$ when n is a power of 2.

Definition 6.1. Let A be a subgroup of $U(n)$. Suppose $S = (x_1, \ldots, x_k)$ and $T = (y_1, \ldots, y_k)$ are sequences in $\mathbb{Z}_n$. We say that S and T are *A-equivalent* if there is a unit $c \in U(n)$, a permutation $\sigma \in S_k$, and $a_1, \ldots, a_k \in A$ such that for each $i \in [1, k]$ we have $c\, y_{\sigma(i)} = a_i x_i$.

Remark 6.2. Let A be a subgroup of $U(n)$. If S is a D-extremal sequence for A, and if S and T are A-equivalent, then T is a D-extremal sequence for A. Also, if S is an E-extremal sequence for A, and if S and T are A-equivalent, then T is an E-extremal sequence for A.

When $n = p^r$ where p is an odd prime, in Theorem 3 of [2] it is shown that a sequence in $\mathbb{Z}_n$ is an E-extremal sequence for $U(n)$ if and only if it is $U(n)$-equivalent to the sequence

$$(\underbrace{0, 0, \ldots, 0}_{n-1 \text{ times}}, 1, p, p^2, \ldots, p^{r-1}).$$

In Theorem 4 of [2] it is shown that when $n = 2^r$, a sequence in $\mathbb{Z}_n$ is a D-extremal sequence for $U(n)$ if and only if it is $U(n)$-equivalent to the sequence $(1, 2, 2^2, \ldots, 2^{r-1})$.

The next result is an immediate consequence of Theorem 5.5.

Theorem 6.3. *For every even positive integer n we have that $E_{U(n)} = n + \Omega(n)$.*

Griffiths and Luca show in [6] and [8] that $E_{U(n)} = n + \Omega(n)$ for every n.

Theorem 6.4. *A sequence in $\mathbb{Z}_{2^r}$ is an E-extremal sequence for $U(2^r)$ if it is $U(2^r)$-equivalent to a sequence of length $n + r - 1$ in which 2^i occurs exactly once for each $i \in [0, r-2]$, and there exists an odd number $m \in [1, 2^r - 1]$, such that 2^{r-1} occurs exactly m times and the remaining terms are zero.*

Proof. Let S be a sequence as in the statement of the theorem. Suppose T is a $U(2^r)$-weighted zero-sum subsequence of S of length 2^r. Let $J = \{\, i \in [0, r-2] : 2^i \in T \,\}$. Suppose $J \neq \emptyset$. Then J has a least element i_0. As T cannot have only one non-zero term, we see that 2^i is a term of T for some $i \in [i_0 + 1, r - 1]$. As 2^{i_0+1} divides all the terms of T except 2^{i_0}, and as T is a $U(2^r)$-weighted zero-sum sequence, we get the contradiction that a unit is divisible by 2.

Thus, we see that $J = \emptyset$. It follows that T is a sequence of length 2^r in which there are an odd number of non-zero terms all of which are equal to 2^{r-1}. As T is a $U(2^r)$-weighted zero-sum sequence, we see that an odd multiple of 2^{r-1} is zero. This gives the contradiction that $2^{r-1} = 0$. Hence, we see that S is a sequence of length $2^r + r - 1$ which does not have any $U(2^r)$-weighted zero-sum subsequence of length 2^r. From Theorem 6.3 we get that $E_{U(2^r)} = 2^r + r$. So we see that S is an E-extremal sequence for $U(2^r)$. It follows that a sequence which is $U(2^r)$-equivalent to S is also an E-extremal sequence for $U(2^r)$. $\square$

The next result is Lemma 1 (ii) of [8].

Lemma 6.5 ([8])**.** *If a sequence in $\mathbb{Z}_{2^r}$ has a non-zero even number of units, then it is a $U(2^r)$-weighted zero-sum sequence.*

Theorem 6.6. *If a sequence in $\mathbb{Z}_{2^r}$ is an E-extremal sequence for $U(2^r)$, then it is $U(2^r)$-equivalent to a sequence in which 2^i occurs exactly once for each $i \in [0, r-2]$, and there is an odd number $m \in [1, 2^r - 1]$ such that 2^{r-1} occurs exactly m times, and the remaining terms are zero.*

Proof. Let $n = 2^r$ and S be a sequence in $\mathbb{Z}_n$ which is an E-extremal sequence for $U(n)$. By Theorem 6.3 we have that $E_{U(n)} = n + r$. So any E-extremal sequence for $U(n)$ has length $n + r - 1$. Suppose S has at least two units. By Lemma 6.5 we see that S has a zero-sum subsequence of length t for any even t which is at most $k - 1$. We may assume that $r \geq 2$, and so we get that $k - 1 = n + r - 2 \geq n$. Thus, we get the contradiction that S has a $U(n)$-weighted zero-sum subsequence of length n. Hence, we see that S has at most one unit.

Let $s \leq r - 2$. Suppose for each $i \in [0, s-1]$, the sequence S has at most one term which is a unit multiple of 2^i. We claim that S has at most one term which is a unit multiple of 2^s. By our assumption, we see that S has at least $k - s$ terms which are divisible by 2^s. If our claim is not true, we can find a subsequence of S having length $k - s - 1$ which has an even number of terms which are a unit multiple of 2^s.

So given any even number t which is at most $k - s - 1$, by Lemma 6.5 we see that S has a $U(n)$-weighted zero-sum subsequence having length t. As $n = k - (r-1)$ and $s \leq r - 2$, it follows that $n \leq k - 1 - s$. Thus, we get the contradiction that S has a $U(n)$-weighted zero-sum subsequence of length n. Hence, our claim must be true. Thus, we see by induction that for each $i \in [0, r-2]$ the sequence S can have at most one term which is a unit multiple of 2^i.

We now claim that for each $i \in [0, r-2]$ the sequence S has exactly one term which is a unit multiple of 2^i. If not, there are at most $r - 2$ such terms, and so S will have at least $k - (r-2) = n + 1$ terms which are either zero or a unit multiple of 2^{r-1}. We can find a subsequence of S having length n which has an even number of terms which are a unit multiple of 2^{r-1}. So by Lemma 6.5 we get the contradiction that S has a $U(n)$-weighted zero-sum subsequence of length n. Hence, we see that our claim is true.

Thus, we see that S has $k - (r-1) = n$ terms which are either zero or a unit multiple of 2^{r-1}. By Lemma 6.5 we see that the number of terms of S which are a unit multiple of 2^{r-1} must be odd. Thus, S is $U(n)$-equivalent to a sequence in which 2^i occurs exactly once for each $i \in [0, r-2]$, and there is an odd number $m \in [1, n-1]$ such that 2^{r-1} occurs exactly m times, and the remaining terms are zero. $\square$

For example, a sequence in $\mathbb{Z}_8$ is an E-extremal sequence for $U(8)$ if and only if it is $U(8)$-equivalent to one of the following sequences:

$$(\,1,\,2,\,4,\,0,\,0,\,0,\,0,\,0,\,0,\,0\,),\;(\,1,\,2,\,4,\,4,\,4,\,0,\,0,\,0,\,0,\,0\,),$$

$$(\,1,\,2,\,4,\,4,\,4,\,4,\,4,\,0,\,0,\,0\,),\;(\,1,\,2,\,4,\,4,\,4,\,4,\,4,\,4,\,4,\,0\,).$$

We will now characterize the C-extremal sequences for $U(2^r)$. The following result is Lemma 5 in [9].

Lemma 6.7 ([9])**.** *Let S be a sequence in $\mathbb{Z}_n$ and p be a prime divisor of n such that every term of S is divisible by p. Suppose $n' = n/p$ and S' is the sequence in $\mathbb{Z}_{n'}$ whose terms are obtained by dividing the terms of S by p and taking their images under $f_{n,n'}$. If S' is a $U(n')$-weighted zero-sum sequence, then S is a $U(n)$-weighted zero-sum sequence.*

If S is a sequence in $\mathbb{Z}_n$ such that all the terms of S are divisible by d, then S/d denotes the sequence in $\mathbb{Z}_n$ whose terms are obtained by dividing the corresponding terms of S by d.

Theorem 6.8. *Let $n = 2^r$ and $n' = n/2$. Suppose $S = (x_1, \ldots, x_{n-1})$ is a sequence in $\mathbb{Z}_n$, $S_1 = (x_1, \ldots, x_{n'-1})$, and $S_2 = (x_{n'+1}, \ldots, x_{n-1})$. Then S is a C-extremal sequence for $U(n)$ if and only if all the terms of S are even except the term $x_{n'}$, and the images of $S_1/2$ and $S_2/2$ under $f_{n,\,n'}$ are C-extremal sequences for $U(n')$.*

Proof. Let $S = (x_1, \ldots, x_{n-1})$ be a C-extremal sequence for $U(n)$. Suppose S has at least two odd terms. We can find a subsequence T of consecutive terms of S which has exactly two odd terms. By Lemma 6.5 we get the contradiction that T is a $U(n)$-weighted zero-sum sequence. Thus, we see that at most one term of S can be odd.

Suppose $x_{n'}$ is even where $n' = n/2$. As at most one term of S is odd, we can find a subsequence T of consecutive terms of S of length n' all of whose terms are even. Let T' be the image of $T/2$ under $f_{n,n'}$. As $n' = 2^{r-1}$ by Theorem 3.6 we have that $C_{U(n')} = n'$. Hence, as T' has length n' we see that T' has an $U(n')$-weighted zero-sum subsequence of consecutive terms. So by Lemma 6.7 we get the contradiction that T (and hence S) has a $U(n)$-weighted zero-sum subsequence of consecutive terms.

Thus, we see that S has a unique odd term which is $x_{n'}$. Let S_1' and S_2' be the images of $S_1/2$ and $S_2/2$ under $f_{n,\,n'}$. Suppose S_1' has a $U(n')$-weighted zero-sum subsequence of consecutive terms. By Lemma 6.7 we get the contradiction that S_1 (and hence S) has a $U(n)$-weighted zero-sum subsequence of consecutive terms. As S_1' has length $n'-1$, it follows that S_1' is a C-extremal sequence for $U(n')$. A similar argument shows that S_2' is also a C-extremal sequence for $U(n')$.

The proof of the converse statement is similar to the proof of Theorem 5 of [10], and hence we omit it. ☐

For example, a sequence S in $\mathbb{Z}_8$ is a C-extremal sequence for $U(8)$ if and only if there exist units $a_1, \ldots, a_7 \in \mathbb{Z}_8$ such that

$$S = (4a_1, 2a_2, 4a_3, a_4, 4a_5, 2a_6, 4a_7).$$

7. Some Other Weight-Sets

For a real number x, we denote the smallest integer which is greater than or equal to x by $\lceil x \rceil$.

Theorem 7.1. *Let $n \geq 2$ be an integer and $r \in [1, n-1]$. Suppose $A = [1, r] \subseteq \mathbb{Z}_n$. Then we have that $C_A = \lceil n/r \rceil$.*

Proof. In Theorem 3 (i) of [15] it is shown that $D_A = \lceil n/r \rceil$. As we have that $C_A \geq D_A$, it follows that $C_A \geq \lceil n/r \rceil$. So it suffices to show that $C_A \leq \lceil n/r \rceil$. Let $S = (x_1, \ldots, x_m)$ be a sequence in $\mathbb{Z}_n$ of length $m = \lceil n/r \rceil$. Consider the sequence

$$S' = (\overbrace{x_1, \ldots, x_1}^{r \text{ times}}, \overbrace{x_2, \ldots, x_2}^{r \text{ times}}, \ldots, \overbrace{x_m, \ldots, x_m}^{r \text{ times}}).$$

We observe that the length of S' is mr which is at least n. By Theorem 1 of [9] we see that $C(\mathbb{Z}_n) \leq n$. So it follows that S' has a zero-sum subsequence of consecutive terms. Thus, we obtain an A-weighted zero-sum subsequence of S having consecutive terms. Hence, it follows that $C_A \leq \lceil n/r \rceil$. □

Theorem 7.2. *Let n be an even number such that $v_2(n) = r$ and let $m = n/2^r$. Suppose B is the set of all the odd elements of $\mathbb{Z}_n$. Then for every $A \subseteq \mathbb{Z}_n$ such that $\{m\} \subseteq A \subseteq B$ we have that $C_A = 2^r$.*

Proof. Let $S = (x_1, \ldots, x_k)$ be a sequence in $\mathbb{Z}_n$ having length $k = 2^r$. Let S' denote the image of the sequence S under the map $f_{n,2^r}$. By applying Theorem 1 of [9] to $\mathbb{Z}_{2^r}$ we see that $C(\mathbb{Z}_{2^r}) \leq 2^r$. So there is a subsequence T of S having consecutive terms such that the image T' of T under $f_{n,2^r}$ is a zero-sum subsequence. It follows that the sum of the terms of T is divisible by 2^r. So we see that T is an $\{m\}$-weighted zero-sum sequence. Thus, we see that $C_{\{m\}} \leq 2^r$.

By Corollary 2 of [9] we have that $C_{U(2^r)} \geq 2^r$. So there is a sequence S' of length $2^r - 1$ in $\mathbb{Z}_{2^r}$ which has no $U(2^r)$-weighted zero-sum subsequence of consecutive terms. As the map $f_{n,2^r}$ is onto, we can find a sequence S in $\mathbb{Z}_n$ whose image under $f_{n,2^r}$ is S'. Suppose S has a B-weighted zero-sum subsequence having consecutive terms. As n is even, we see that the image of B under $f_{n,2^r}$ is contained in $U(2^r)$. So we get the contradiction that S' has a $U(2^r)$-weighted zero-sum subsequence having consecutive terms. Thus, we see that $C_B \geq 2^r$.

As $\{m\} \subseteq A \subseteq B$, we see that $C_B \leq C_A \leq C_{\{m\}}$. Hence, it follows that $C_A = 2^r$. □

Our next result generalizes a result of [15]. We follow a similar argument as in Theorem 7.2.

Theorem 7.3. *Let n be an even number such that $v_2(n) = r$ and let $m = n/2^r$. Suppose B denotes the set of all the odd elements of $\mathbb{Z}_n$. Then for every $A \subseteq \mathbb{Z}_n$ such that $\{m\} \subseteq A \subseteq B$ we have that $D_A = r + 1$.*

Proof. Let $S = (x_1, \dots, x_k)$ be a sequence in $\mathbb{Z}_n$ having length $k = r + 1$. Let S' denote the image of the sequence S under the map $f_{n,2^r}$. By Theorem 4.4 we have that $D_{U(2^r)} = r + 1$. So there is a subsequence T of S such that the image T' of T under $f_{n,2^r}$ is a $U(2^r)$-weighted zero-sum subsequence. By Lemma 7 of [10] we see that the map $f_{n,2^r}$ maps $U(n)$ onto $U(2^r)$. So if I denotes the set $\{i : x_i \text{ is a term of } T\}$, we see that for each $i \in I$ there exists $a_i \in U(n)$ such that $f_{n,2^r}\big(\sum_{i\in I} a_i x_i\big) = 0$. Hence, it follows that $\sum_{i\in I} a_i x_i$ is divisible by 2^r, and so we see that T is an $\{m\}$-weighted zero-sum sequence. Thus, we see that $D_{\{m\}} \leq r+1$.

By Theorem 4.4 we have that $D_{U(2^r)} = r+1$. So there is a sequence S' of length r in $\mathbb{Z}_{2^r}$ which has no $U(2^r)$-weighted zero-sum subsequence. The rest of the proof is very similar to the argument given in the second and third paragraphs of the proof of Theorem 7.2. Hence, we get that $D_A = r + 1$. □

Remark 7.4. When n is even and A is the set of all even elements of $\mathbb{Z}_n$, in [15] it is shown that $D_A = 2$. So it follows that $C_A = 2$.

When n is odd and A is the set of all odd (or all even) elements of $\mathbb{Z}_n$, in [15] it is shown that $D_A = 3$. From the proof of this result in [15] we see that $C_A = 3$.

Theorem 7.5. *Let k be an odd prime and p be a prime such that $p \equiv 1 \pmod{k}$ and $p \not\equiv 1 \pmod{k^2}$. Then we have that $D_{U(p)^k} \leq k$.*

Proof. As $p \equiv 1 \pmod{k}$ there exists $c \in U(p)$ such that c has order k. The subgroup $U(p)^k$ is the image of the map $U(p) \to U(p)$ given by $x \mapsto x^k$. The kernel of this map has at most k elements and contains c. As c has order k, it follows that the kernel is $\langle c \rangle$. Thus, we have that $|\,U(p)\,| = k\,|U(p)^k|$ and so we see that $U(p)/U(p)^k$ has order k.

Suppose $c \in U(p)^k$. Then there exists $a \in U(p)$ such that $c = a^k$, and so $a^{k^2} = c^k = 1$. As k is a prime and $a^k = c \neq 1$, we see that the order of a is k^2. However, as $p \not\equiv 1 \pmod{k^2}$ there is no element of order k^2 in $U(p)$. Thus, we see that $c \notin U(p)^k$. For $x \in U(p)$ if we denote the coset $xU(p)^k$ by $[x]$ we see that $[c] \neq [1]$. As k is prime, we see that $[c]$ has order k. Hence, we get a partition of $U(p)$ by the cosets $[1]$, $[c]$, …, $[c^{k-1}]$.

Let $S = (x_1, \dots, x_k)$ be a sequence in $U(p)$. Suppose we show that S has a $U(p)^k$-weighted zero-sum subsequence. It will follow that $D_{U(p)^k} \leq k$.

`Case 1:` There exist two elements of S which are in the same coset.

As there exists $l \in [0, k-1]$ such that $x_i, x_j \in [c^l]$, it follows that there exist $a, b \in U(p)^k$ such that $x_i = a\,c^l$ and $x_j = b\,c^l$. Then we have that $(-b)\,x_i + a\,x_j = 0$. As k is odd, we see that $-1 \in U(p)^k$. Hence, it follows that (x_i, x_j) is a $U(p)^k$-weighted zero-sum subsequence of S.

`Case 2:` No two elements of S are in the same coset.

Without loss of generality, we can assume that for each $i \in [1, k]$ there exist $a_i \in U(p)^k$ such that $x_i = a_i\,c^{i-1}$. Then we see that $a_1^{-1}\,x_1 + a_2^{-1}\,x_2 + \cdots + a_k^{-1}\,x_k =$

$1 + c + \cdots + c^{k-1} = 0$ as c satisfies $X^k - 1 = (X-1)\,(X^{k-1} + \cdots + X + 1)$ and $c \neq 1$. Thus, it follows that S is a $U(p)^k$-weighted zero-sum sequence. □

From Theorem 3 of [3] we see that $D_{U(p)^2} = 3$ where p is an odd prime. This shows that Theorem 7.5 is not true when $k = 2$. The next result is Corollary 1 of [10].

Lemma 7.6 ([10])**.** *Let F be a field and A be a subgroup of F^*. A sequence $S = (x, y)$ in F does not have an A-weighted zero-sum subsequence if and only if x and $-y$ are in different cosets of A in F^*.*

Corollary 7.7. *Let F be a field and A be a proper subgroup of F^*. Then we have that $D_A \geq 3$.*

Corollary 7.8. *Let $k \geq 2$ be an integer and p be a prime such that $p \equiv 1 \pmod{k}$. Then we have that $D_{U(p)^k} \geq 3$.*

Proof. As $p \equiv 1 \pmod{k}$, there is an element in $U(p)$ of order k. So as in the first paragraph of the proof of Theorem 7.5, we see that the index of the subgroup $U(p)^k$ of $U(p)$ is k. As $k \geq 2$, we see that $U(p)^k$ is a proper subgroup of $U(p)$, and so the result follows from Corollary 7.7. □

8. Concluding Remarks

From Remark 7.4 we see that when $A = \{a \in [1, n-1] : a \text{ is odd}\}$ and n is odd, we have that $C_A = D_A = 3$. Also, from Theorem 3 of [3] and Theorem 4 of [9], we see that $C_{Q_p} = D_{Q_p} = 3$ when p is an odd prime. This makes us curious to see if we can find a weight-set $A \subseteq \mathbb{Z}_n$ when n is odd, such that A has size $(n-1)/2$ and either $D_A > 3$ or $C_A > 3$. It will also be interesting to characterize the C-extremal and D-extremal sequences for $U(n)$ when n is an even number which is not a power of 2.

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