ALGORITHMS AND BOUNDS ON THE SUMS OF POWERS OF CONSECUTIVE PRIMES

Cathal O'Sullivan[1]
Department of Mathematics, Statistics, and Actuarial Science, and Department of Computer Science and Software Engineering, Butler University, Indianapolis, Indiana

Jonathan P. Sorenson
Department of Computer Science and Softare Engineering, Butler University, Indianapolis, Indiana
jsorenso@butler.edu

Aryn Stahl[2]
Department of Physics and Astronomy, and Department of Computer Science and Software Engineering, Butler University, Indianapolis, Indiana

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Abstract

For an integer $k > 1$, let $s_k(x)$ count the number of representations of integers $n \leq x$ as the sum of kth powers of consecutive primes. We present and analyze an algorithm to enumerate all such integers n and an algorithm to compute the value of $s_k(x)$. We also present asymptotic upper and lower bounds on $s_k(x)$ that are within a constant factor of one another. In particular, we show that $s_k(x) \sim x^{2/(k+1)+o(1)}$. This work extends previous work by Tongsomporn, Wananiyakul, and Steuding (2022) who examined sums of squares of consecutive primes.

1. Introduction

Let $\mathcal{S}_k(x)$ denote the set of integers $n \leq x$ that can be written as a sum of the kth powers of consecutive primes. For example, $5^3 + 7^3 + 11^3 = 1799$ is an element of $\mathcal{S}_3(2000)$. Let $s_k(x)$ be the number of such n, counted with multiplicity. If a specific integer n has more than one representation as the sum of kth powers of consecutive primes, we count all such representations when we say "with multiplicity". So we have $s_k(x) \geq \#\mathcal{S}_k(x)$.

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[1]Current affiliation: AbbVie, Lake Bluff, Illinois

[2]Current affiliation: Naval Surface Warfare Center, Crane, Indiana

In this paper, we describe an algorithm that, given k and x, produces the elements of $\mathcal{S}_k(x)$ along with their representation. Its running time is linear in $s_k(x)$, the number of such representations. The algorithm uses $O(kx^{1/k})$ space. This is Section 2. In Section 3, we describe a second algorithm that computes the value of $s_k(x)$, with multiplicity, given k and x. This algorithm takes $O(x^{1/k}/\log\log x)$ arithmetic operations, the time it takes to find all primes up to $x^{1/k}$. In Section 4 we show that

$$s_k(x) \leq (1+o(1)) \cdot c_k \frac{x^{2/(k+1)}}{(\log x)^{2k/(k+1)}},$$

where $c_k = (k^2/(k-1)) \cdot (k+1)^{1-1/k}$. This is a generalization of a bound for $s_2(x)$ proven in [6]. Their bound is explicit and ours is not. This is also an upper bound on the number of arithmetic operations used by our enumeration algorithm. Also in Section 4, we give the lower bound

$$s_k(x) \geq (1+o(1)) \cdot \frac{(k+1)^2}{2} \frac{x^{2/(k+1)}}{(\log x)^{2k/(k+1)}},$$

In Section 5 we apply our new algorithm to compute $\mathcal{S}_k(x)$ for various x and k, and give some examples of integers that can be written as sums of consecutive powers of primes in more than one way. Note that $\mathcal{S}_2(5000)$ was computed by [6]; see also sequence A340771 at the On-Line Encyclopedia of Integer Sequences [1].

We begin by describing our enumeration algorithm in the next section.

2. The Enumeration Algorithm

Given as input a bound x and integer exponent $k > 1$, our algorithm produces the elements of the set $\mathcal{S}_k(x)$ as follows.

Let $p_1 = 2, p_2 = 3, \ldots$ denote the primes, and let $\pi(y)$ denote the number of primes less than or equal to y. By the Prime Number Theorem (see, for example, [4]), $\pi(y) \sim y/\log y$, and thus $p_\ell \sim \ell \log \ell$.

We assume all arithmetic operations take constant time. In practice, all our integers are at most 128 bits, or roughly 38 decimal digits.

The three steps of our algorithm are as follows:

1. Find the primes up to $x^{1/k}$.
2. Compute the prefix sum array $f[\,]$, where $f[0] = 0$ and $f[i] := p_1^k + p_2^k + \cdots + p_i^k$ for all $i \leq \pi(x^{1/k})$, so that $f[i+1] = f[i] + p_{i+1}^k$.
3. Loops to enumerate $\mathcal{S}_k(x)$:

for $b := 0$ to $\pi(x^{1/k}) - 1$ do:

for $t := b+1$ to $\pi(x^{1/k})$ do:
  $n := f[t] - f[b]$;
  if $n > x$ break the t loop,
  else output(n, p_{b+1});

Remark 1. Step 1 is not the bottleneck, so the Sieve of Eratosthenes is sufficient, taking $O(x^{1/k}\log\log x)$ time. See also [2, 5]. Note that in the second step, the value of the largest entry in the array is bounded by $x^{1+1/k}$. If we use a binary algorithm for integer exponentiation, Step 2 takes time $O(\pi(x^{1/k})\log k)$, which is smaller than the asymptotic bound given for Step 1. Storing $f[\,]$ uses $O(kx^{1/k})$ bits of space. The time for Step 3 is proportional to the number of (n, p_{b+1}) pairs that are output, which is $s_k(x)$. This, in turn, we bound asymptotically in Theorem 1 below, at $c_k x^{2/(k+1)}/(\log x)^{2k/(k+1)}$ time. We output pairs (n, p_{b+1}) in case a specific value of n gets repeated. If we have repeats for n, the p_{b+1} values will be different, and p_{b+1} is the first prime in the sequence of powers of primes to generate n, allowing us to quickly reconstruct two (or more) representations of n as kth powers of consecutive primes. In practice, we found repeated values of n to be quite rare.

Example 1. Let us compute $\mathcal{S}_3(1000)$.

1. We find the primes up to $1000^{1/3} = 10$, so $2, 3, 5, 7$.

2. We compute the prefix array $f[\,]$ as follows:

0	1	2	3	4
0	8	35	160	503

3. We generate the $f[t] - f[b]$ values, and hence $\mathcal{S}_3(1000)$, as follows:

$$b = 0 : (8, 2), (35, 2), (160, 2), (503, 2)$$
$$b = 1 : (27, 3), (152, 3), (495, 3)$$
$$b = 2 : (125, 5), (468, 5)$$
$$b = 3 : (343, 7)$$

   And thus $s_3(1000) = 10$.

3. A Counting Algorithm

In this section we describe an algorithm that computes $s_k(x)$, with multiplicity. Since we are not explicitly constructing all the representations of the integers, we are able to save a considerable amount of time by collapsing the inner loop in Step 3 of the previous algorithm.

1. Find the primes up to $x^{1/k}$.

2. Compute the prefix sum array $f[\,]$ as done above.

3. Loop to compute the count:

   $count := 0$; $t := 0$;
   while($t < \pi(x^{1/k})$ and $f[t+1] \le x$) do:
   $\quad t := t + 1$;
   for $b := 0$ to $\pi(x^{1/k}) - 1$ do:
   $\quad$ if $t < \pi(x^{1/k})$ and $f[t+1] - f[b] \le x$ then
   $\quad\quad t := t + 1$;
   $\quad count := count + (t - b)$;

It is easy to see that the running time for the last step is $O(\pi(x^{1/k}))$ arithmetic operations. So finding the primes in Step 1 dominates the running time, at $O(x^{1/k}/\log\log x)$ time using, say, the Atkin-Bernstein prime sieve [2].

4. Analysis

In this section we prove the following theorem, which provides an upper bound on $s_k(x)$.

Theorem 1. *For $k > 1$ we have*

$$s_k(x) \le (1 + o(1)) \cdot c_k \frac{x^{2/(k+1)}}{(\log x)^{2k/(k+1)}},$$

where $c_k = (k^2/(k-1)) \cdot (k+1)^{1-1/k}$.

Note that $c_k \sim k^2$ for large k. In [6] the authors prove the explicit bound

$$s_2(x) \le 28.4201 \frac{x^{2/3}}{(\log x)^{4/3}}.$$

We also have the trivial lower bound $s_k(x) \ge \pi(x^{1/k}) \sim kx^{1/k}/\log x$ by the Prime Number Theorem.

Our proof follows the same lines as in [6]. We begin by partitioning the members of $\mathcal{S}_k(x)$ by the number of prime powers m in their representative sum. Define

$$s_{k,m}(x) = \#\{n \le x : \text{there exists } \ell \ge 0 : n = p_{\ell+1}^k + \cdots + p_{\ell+m}^k\}$$

so that $s_k(x) = \sum_{m=1}^{M} s_{k,m}(x)$ for a sufficiently large, and as yet unknown value $M = M(x,k)$, the length of the longest sum of powers of consecutive primes less that or equal to x.

Lemma 1 ([6])**.** *For $k > 1$ and $m > 0$ we have*

$$s_{k,m}(x) \leq \pi((x/m)^{1/k}).$$

Proof. Let $\ell = s_{k,m}(x)$. We have

$$mp_\ell^k \leq p_\ell^k + p_{\ell+1}^k + \cdots + p_{\ell+(m-1)}^k \leq x.$$

Thus $mp_\ell^k \leq x$, or $p_\ell \leq (x/m)^{1/k}$, or $s_{k,m}(x) = \ell \leq \pi((x/m)^{1/k})$. □

Next, we need an estimate for M.

Lemma 2. *For $k > 1$ we have*

$$M(x,k) \sim (k+1)\frac{x^{1/(k+1)}}{(\log x)^{k/(k+1)}}.$$

Proof. We have

$$\sum_{\ell=1}^{M} p_\ell^k \leq x < \sum_{\ell=1}^{M+1} p_\ell^k.$$

Using the asymptotic estimate $p_M \sim M \log M$ from the Prime Number Theorem and using the methods from [3, Section 2.7] we have

$$\begin{aligned} \sum_{\ell=1}^{M} p_\ell^k &\sim \sum_{p \leq M \log M} p^k = \int_2^{M \log M} t^k d\pi(t) \\ &\sim \int_2^{M \log M} \frac{t^k}{\log t} dt \sim \frac{1}{\log M} \int_2^{M \log M} t^k dt, \end{aligned}$$

and so we have

$$x \sim \frac{(M \log M)^{k+1}}{(k+1)\log M}.$$

Taking the logarithm of both sides gives us $(k+1)\log M \sim \log x$. We then obtain that

$$M \sim (k+1)(x \log x)^{1/(k+1)}/\log x.$$

□

We are now ready to prove Theorem 1.

Proof. We have

$$s_k(x) = \sum_{m=1}^{M} s_{k,m}(x) \leq \sum_{m=1}^{M} \pi((x/m)^{1/k}).$$

By the Prime Number Theorem and our lemmas, we have

$$\begin{aligned} s_k(x) &\leq \sum_{m=1}^{M} \pi((x/m)^{1/k}) \sim \sum_{m=1}^{M} k(x/m)^{1/k}/\log(x/m) \\ &\sim \frac{kx^{1/k}}{\log x}\sum_{m=1}^{M} m^{-1/k} \sim \frac{kx^{1/k}}{\log x}\frac{M^{1-1/k}}{1-1/k}. \end{aligned}$$

Plugging in our estimate for M from Lemma 2 gives this bound for $s_k(x)$:

$$\frac{kx^{1/k}}{\log x}\frac{k}{k-1}\left((k+1)\frac{x^{1/(k+1)}}{(\log x)^{k/(k+1)}}\right)^{1-1/k}.$$

A bit of algebra simplifies the exponents to complete the proof. □

Note that $\lim_{k\to\infty}(1/k)\cdot(k+1)^{1-1/k} = 1$.

We wrap up this section with our lower bound proof for $s_k(x)$.

Theorem 2. *For $k > 1$ we have*

$$s_k(x) \geq \binom{M}{2} \geq (1+o(1)) \cdot \frac{(k+1)^2}{2}\frac{x^{2/(k+1)}}{(\log x)^{2k/(k+1)}}.$$

Proof. Consider the sum $p_1^k + \cdots + p_M^k$, which is at most x by definition. The lower bound is obtained by counting the number of i, j pairs with $1 \leq i \leq j \leq M$, which is $\binom{M}{2}$, as each sum $p_i^k + \cdots + p_j^k$ gives an integer n counted by $s_k(x)$. Thus, $s_k(x) \geq M(M-1)/2$, and apply Lemma 2. □

5. Empirical Results

In this section we give some of our empirical results. This is not everything we have – the interested reader is encouraged to contact the second author for copies of the data or source code.

5.1. Tightness of Theorems 1 and 2

In Tables 1–5, we present values of $s_k(x)$ for $k = 2, 3, 5, 10, 20$ for x up to 10^{38}, which is close to the limit for 128-bit hardware integer arithmetic. We also include the upper bound from Theorem 1 and the lower bound from Theorem 2.

Following the tables, we have graphed our upper bound (green) and lower bound (blue) estimates with the exact counts (purple) from Tables 1–5 to show how tight our bounds are in practice. In our graph for $k = 2$ we also include the explicit upper bound (orange) from [6].

x	$s_2(x)$	Upper	Lower
10^3	37	52	34
10^4	132	166	108
10^5	519	574	372
10^6	1998	2089	1357
10^7	7840	7898	5130
10^8	31372	30681	19928
10^9	126689	121714	79056
10^{10}	517191	490907	318853
10^{11}	2132474	2006670	1303370
10^{12}	8867094	8293885	5387036
10^{13}	37153225	34599930	22473314
10^{14}	156713533	145488607	94497622
10^{15}	665005737	615948906	400070550

Table 1: Values of $s_2(x)$, with upper and lower bounds

x	$s_3(x)$	Upper	Lower
10^3	10	19	13
10^4	29	40	28
10^5	70	91	64
10^6	186	220	155
10^7	491	554	390
10^8	1297	1434	1011
10^9	3501	3801	2681
10^{10}	9568	10262	7240
10^{11}	26429	28130	19846
10^{12}	73575	78071	55080
10^{13}	206617	218951	154472
10^{14}	584184	619541	437093
10^{15}	1663904	1766547	1246320
10^{16}	4769563	5070868	3577556
10^{17}	13742399	14641613	10329827
10^{18}	39796129	42496537	29981799
10^{19}	115807012	123917289	87425082
10^{20}	338386013	362841801	255989092

Table 2: Values of $s_3(x)$, with upper and lower bounds

x	$s_5(x)$	Upper	Lower
10^5	10	20	14
10^6	21	32	22
10^7	38	54	37
10^8	68	94	65
10^9	127	167	115
10^{10}	243	302	208
10^{11}	479	556	382
10^{12}	862	1037	712
10^{13}	1639	1956	1343
10^{14}	3128	3725	2558
10^{15}	6053	7154	4913
10^{16}	11799	13841	9507
10^{17}	22938	26954	18513
10^{18}	44869	52794	36262
10^{19}	87959	103940	71393
10^{20}	173621	205585	141209
10^{21}	343199	408328	280466
10^{22}	681611	814086	559167
10^{23}	1359330	1628652	1118664
10^{24}	2717318	3268557	2245058
10^{25}	5451410	6578721	4518694
10^{26}	10962586	13276572	9119214
10^{27}	22107170	26859747	18449024
10^{28}	44656828	54464244	37409592
10^{29}	90459929	110673813	76017986
10^{30}	183613129	225340599	154778606
10^{31}	373421607	459662117	315725893
10^{32}	761023562	939272425	645153503

Table 3: Values of $s_5(x)$, with upper and lower bounds

x	$s_{10}(x)$	Upper	Lower
10^{10}	10	21	13
10^{11}	15	26	16
10^{12}	21	35	22
10^{13}	36	45	28
10^{14}	45	61	38
10^{15}	56	81	51
10^{16}	78	110	69
10^{17}	120	150	94
10^{18}	154	206	129
10^{19}	214	284	178
10^{20}	301	393	247
10^{21}	439	547	344
10^{22}	599	765	481
10^{23}	832	1072	674
10^{24}	1187	1508	949
10^{25}	1678	2129	1339
10^{26}	2373	3013	1895
10^{27}	3304	4276	2690
10^{28}	4817	6083	3827
10^{29}	6786	8674	5457
10^{30}	9744	12396	7799
10^{31}	13788	17751	11168
10^{32}	19871	25467	16022
10^{33}	28290	36601	23027
10^{34}	40949	52692	33150
10^{35}	58459	75976	47799
10^{36}	84393	109711	69023
10^{37}	121302	158647	99810
10^{38}	175797	229717	144523

Table 4: Values of $s_{10}(x)$, with upper and lower bounds

x	$s_{20}(x)$	Upper	Lower
10^{20}	10	20	12
10^{21}	15	23	13
10^{22}	15	26	15
10^{23}	21	30	17
10^{24}	21	35	20
10^{25}	28	40	23
10^{26}	36	46	27
10^{27}	36	54	31
10^{28}	45	63	36
10^{29}	45	73	42
10^{30}	66	85	49
10^{31}	66	100	58
10^{32}	78	117	68
10^{33}	105	138	80
10^{34}	120	162	94
10^{35}	136	191	111
10^{36}	171	225	131
10^{37}	190	266	154
10^{38}	232	315	183

Table 5: Values of $s_{20}(x)$, with upper and lower bounds

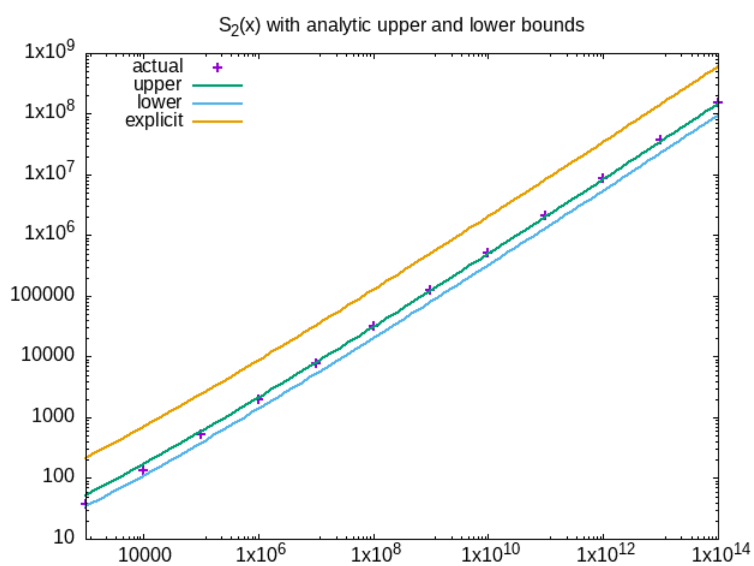

Figure 1: Graph of Table 1

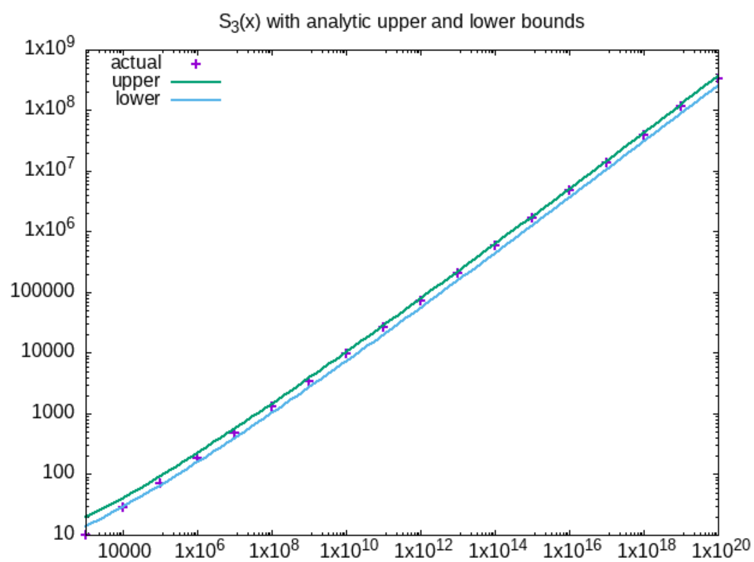

Figure 2: Graph of Table 2

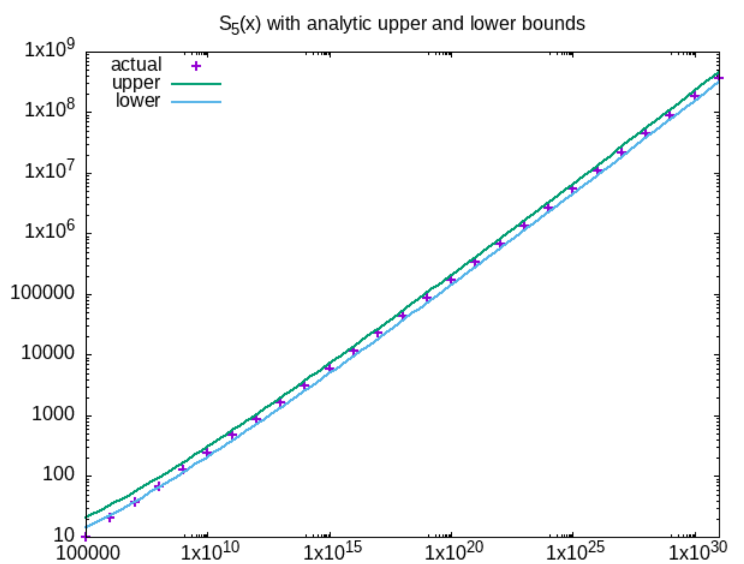

Figure 3: Graph of Table 3

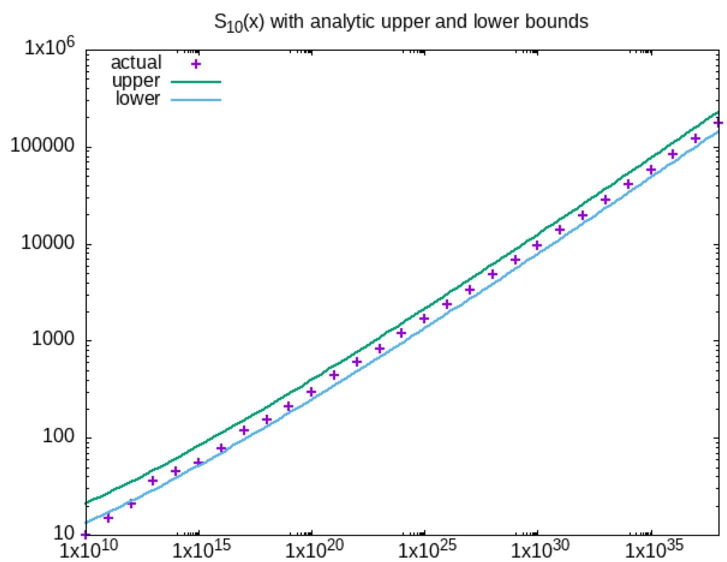

Figure 4: Graph of Table 4

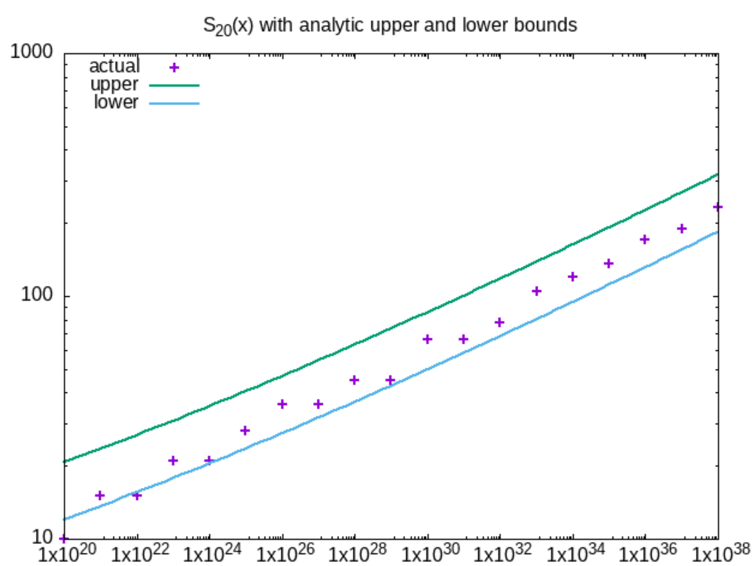

Figure 5: Graph of Table 5

We conclude this subsection with a comparison of the constants c_k from Theorem 1 with the value of $(k+1)^2/2$ from Theorem 2.

k	c_k	$(k+1)^2/2$
2	6.92	4.5
3	11.33	8.0
4	17.83	12.5
5	26.21	18.0
6	36.44	24.5
7	48.54	32.0
8	62.52	40.5
9	78.39	50.0
10	96.16	60.5
11	115.84	72.0
12	137.43	84.5
13	160.94	98.0
14	186.38	112.5
15	213.75	128.0
16	243.05	144.5
17	274.29	162.0
18	307.47	180.5
19	342.60	200.0
20	379.68	220.5

5.2. Duplicates

We found 40 values of $n \le x = 10^{12}$ that have multiple representations as sums of consecutive squares of primes. The smallest such number is 14720439, which can be written as

$$941^2+947^2+953^2+967^2+971^2+977^2+983^2+991^2+997^2+1009^2+ 1013^2+1019^2+1021^2+1031^2+1033^2$$

and as

$$131^2+137^2+139^2+149^2+151^2+157^2+163^2+167^2+173^2+179^2+ 181^2+191^2+193^2+197^2+199^2+211^2+223^2+227^2+229^2+233^2+239^2+ 241^2+251^2+257^2+263^2+269^2+271^2+277^2+281^2+283^2+293^2+307^2+ 311^2+313^2+317^2+331^2+337^2+347^2+349^2+353^2+359^2+367^2+373^2+ 379^2+383^2+389^2+397^2+401^2+409^2+419^2+421^2+431^2+433^2+439^2+ 443^2+449^2+457^2+461^2+463^2+467^2+479^2+487^2+491^2+499^2+503^2+ 509^2+521^2+523^2+541^2+547^2+557^2+563^2+569^2+571^2+577^2+587^2+ 593^2+599^2+601^2+607^2+613^2+617^2+619^2+631^2+641^2+643^2+647^2.$$

To find these, we sorted the output of our algorithm from Section 2, and then used the `uniq -D` unix/linux command to list the duplicates.

We found no integers that can be written as the sum of consecutive powers of primes in more than one way for any power larger than 2. We searched for cubes up to 10^{18}, fifth powers up to 10^{27}, and tenth and twentieth powers up to 10^{38}. This

n	Prime 1	Prime 2	n	Prime 1	Prime 2
14720439	131	941	47638558043	28097	65731
16535628	1123	569	50195886916	479	6857
34714710	2389	401	50811319931	2039	21283
40741208	131	653	56449248367	2803	4127
61436388	569	809	86659250142	4561	53609
603346308	401	919	105146546059	29587	6599
1172360113	3701	4673	119789313426	31847	42299
1368156941	1367	16519	125958414196	16763	26183
1574100889	3623	613	134051910100	183047	4397
1924496102	11657	2803	159625748030	1367	3301
1989253499	3359	613	169046403821	183829	19717
2021860243	3701	4297	263787548443	47297	62347
6774546339	11273	47513	330881994258	11161	2039
9770541610	1663	7243	438882621700	16763	20369
12230855963	10177	2777	507397251905	643	75013
12311606487	28603	3257	572522061248	18427	44371
12540842446	11087	479	687481319598	16139	338461
14513723777	1663	6323	780455791261	3257	7057
26423329489	1709	32401	847632329089	184003	7523
38648724198	2777	6967	854350226239	14821	6599

Table 6: Duplicates for $k = 2$

search requires computing $\mathcal{S}_k(x)$ and not just $s_k(x)$; note that it is much faster to compute just $s_k(x)$ in practice because outputting the elements of $\mathcal{S}_k(x)$ to a text file slows down the computation considerably.

We found exactly one example with differing powers:

$$\begin{aligned} 23939 &= 23^2 + 29^2 + 31^2 + 37^2 + 41^2 + 43^2 + 47^2 + 53^2 + 59^2 + 61^2 + 67^2 \\ &= 17^3 + 19^3 + 23^3. \end{aligned}$$

We conclude this subsection with the list of 40 integers up to 10^{12} that can be written as sums of squares of consecutive primes in two ways. For each such integer in the Table 6, we list the starting primes for each of their two ways to sum.

5.3. Initial Elements of $\mathcal{S}_k$

We wrap up the presentation of our computations with the first few elements of each of the $\mathcal{S}_k$ sets we computed. We have the following:

$\mathcal{S}_2$: 4 9 13 25 34 38 49 74 83 87 121 169 170 195 204 208 289 290 339 361;

$\mathcal{S}_3$: 8 27 35 125 152 160 343 468 495 503 1331 1674 1799 1826 1834 2197 3528 3871 3996 4023;

$\mathcal{S}_5$: 32 243 275 3125 3368 3400 16807 19932 20175 20207 161051 177858 180983 181226 181258 371293 532344 549151 552276 552519;

$\mathcal{S}_{10}$: 1024 59049 60073 9765625 9824674 9825698 282475249 292240874 292299923 292300947;

$\mathcal{S}_{20}$: 1048576 3486784401 3487832977 95367431640625 95370918425026 95370919473602 79792266297612001 79887633729252626 79887637216037027 79887637217085603.

6. Future Work

We have several ideas for future work:

Our primary goal is to parallelize our algorithm from Section 2 to extend our computations. For larger powers, this will also mean using multiple-precision integer arithmetic using, for example, GMP.

More careful proofs of Theorems 1 and 2 might give explicit bounds, or perhaps an asymptotic constant. If such a constant exists, it appears to be near 0.6.

Is there a power $k > 2$ for which there are integers with multiple representations as sums of powers of consecutive primes? We have not found any as of yet.

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