GENERALIZED DEGENERATE HARMONIC NUMBERS AND THEIR APPLICATIONS WITH RIORDAN ARRAYS

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Abstract

In this paper, we give some identities involving degenerate harmonic numbers and some special degenerate numbers by using Riordan arrays. For example,

$$\sum_{k=1}^{n} \frac{(-1)^k \beta_{k,\lambda} H_\lambda(n, k-1, \alpha)}{k!} = \alpha H_{n+1,\lambda}(\alpha) - 1 \quad \text{for } n \geq 1,$$

and

$$\sum_{k=0}^{n} (-1)^k H_{n-k+1,\lambda}^{k+1}(\alpha) = \frac{(-1)^n D_{n,\lambda}}{\alpha^{n+1} n!} \quad \text{for } n \geq 0,$$

where $H_{n,\lambda}(\alpha)$ are generalized degenerate harmonic numbers, $H_{n,\lambda}^r(\alpha)$ are generalized degenerate hyperharmonic numbers of order r, $H_\lambda(n, r, \alpha)$ are generalized degenerate harmonic numbers of rank r, $D_{n,\lambda}$ are degenerate Daehee numbers, and $\beta_{n,\lambda}$ are degenerate Bernoulli numbers.

1. Introduction

For a positive real number λ, the *degenerate exponential functions* ([9], [10], [11], [12], and [15]) are defined by

$$e_\lambda^x(t) = (1 + \lambda t)^{x/\lambda} = \sum_{n=0}^{\infty} (x)_{n,\lambda} \frac{t^n}{n!}, \tag{1}$$

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where $(x)_{0,\lambda} = 1$ and $(x)_{n,\lambda} = x(x-\lambda)\dots(x-(n-1)\lambda)$ for $n \geq 1$. When $x = 1$, it is seen that $e_\lambda^1(t) = e_\lambda(t)$.

Let $\log_\lambda t$ be the compositional inverse function of $e_\lambda(t)$ such that $\log_\lambda(e_\lambda(t)) = e_\lambda(\log_\lambda t) = t$. The $\log_\lambda t$ are called the *degenerate logarithm functions* and are given by

$$\log_\lambda t = \frac{t^\lambda - 1}{\lambda} = \sum_{n=1}^{\infty} \frac{\lambda^{n-1}(1)_{n,1/\lambda}}{n!}(t-1)^n = \sum_{n=1}^{\infty} \frac{1}{\lambda}\binom{\lambda}{n}(t-1)^n, \tag{2}$$

where

$$\binom{x}{k} = \frac{(x)_k}{k!}$$

for a non-negative integer k, a real number x, and $(x)_n = (x)_{n,1}$. Note that $\lim_{\lambda\to 0} \log_\lambda t = \log t$ and $\lim_{\lambda\to 0} e_\lambda(t) = e^t$.

For a non-negative integer r, it is well known that

$$\frac{1}{(1-t)^{r+1}} = \sum_{n=0}^{\infty} \binom{n+r}{n} t^n. \tag{3}$$

Harmonic numbers and their generalizations are important in various branches of combinatorics, number theory, and there has been a lot of work involving these numbers (see [3], [4], [5], [6], [7], [16], and [24]). The *harmonic numbers*, denoted by H_n, are defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for } n \geq 1.$$

For a positive real number α, the *generalized harmonic numbers of rank* r ([6], [20]) denoted by $H(n, r, \alpha)$, are given by

$$H(n,r,\alpha) = \sum_{1 \leq k_0+k_1+\dots+k_r \leq n} \frac{1}{k_0 k_1 \dots k_r \alpha^{k_0+k_1+\dots+k_r}} \quad \text{for } n > r \geq 0.$$

Note that $H(n, r, \alpha) = 0$ for $n \leq r$ by convention. When $\alpha = 1$, $H(n, r, \alpha) = H(n, r)$ were introduced in ([7], [21]).

The *degenerate harmonic numbers* [18] are defined by

$$H_{0,\lambda} = 0 \quad \text{and} \quad H_{n,\lambda} = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{\lambda}\binom{\lambda}{k} \quad \text{for } n \geq 1,$$

and the generating function of these numbers is given by

$$\frac{-\log_\lambda(1-t)}{1-t} = \sum_{n=1}^{\infty} H_{n,\lambda} t^n.$$

Note that $\lim_{\lambda\to 0} H_{n,\lambda} = H_n$.

In [16], Kim et al. introduced the *degenerate hyperharmonic numbers*, denoted by $H^r_{n,\lambda}$, which are given by

$$H^r_{0,\lambda} = 0, \quad H^1_{n,\lambda} = H_{n,\lambda} \quad \text{and} \quad H^r_{n,\lambda} = \sum_{k=1}^{n} H^{r-1}_{k,\lambda} \quad \text{for } n \geq 1, r \geq 2.$$

Also, they gave the generating function of these numbers as

$$\frac{-\log_\lambda(1-t)}{(1-t)^r} = \sum_{n=1}^{\infty} H^r_{n,\lambda} t^n. \tag{4}$$

They investigated some properties, recurrence relations and identities involving degenerate harmonic numbers and degenerate hyperharmonic numbers. For non-negative integers n and k,

$$H^{k+1}_{n,\lambda} = \frac{(-1)^k}{\binom{\lambda-1}{k}} \binom{n+k}{n} (H_{n+k,\lambda} - H_{k,\lambda}).$$

For a positive real number α, Dağlı [4] defined the *generalized degenerate harmonic numbers*, denoted by $H_{n,\lambda}(\alpha)$, as

$$H_{0,\lambda}(\alpha) = 0 \quad \text{and} \quad H_{n,\lambda}(\alpha) = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{\lambda \alpha^k} \binom{\lambda}{k} \quad \text{for } n \geq 1,$$

and the *generalized degenerate hyperharmonic numbers of order* r, denoted by $H^r_{n,\lambda}(\alpha)$, as

$$H^r_{0,\lambda}(\alpha) = 0, \; H^1_{n,\lambda}(\alpha) = H_{n,\lambda}(\alpha) \quad \text{and} \quad H^r_{n,\lambda}(\alpha) = \sum_{k=1}^{n} H^{r-1}_{k,\lambda}(\alpha) \quad \text{for } n \geq 1, \; r \geq 2,$$

with $H^0_{n,\lambda}(\alpha) = \frac{(-1)^{n-1}}{\lambda \alpha^n}\binom{\lambda}{n}$. The generating functions of these numbers are

$$\frac{-\log_\lambda\left(1 - \frac{t}{\alpha}\right)}{1-t} = \sum_{n=0}^{\infty} H_{n,\lambda}(\alpha) t^n,$$

and

$$\frac{-\log_\lambda(1 - \frac{t}{\alpha})}{(1-t)^r} = \sum_{n=0}^{\infty} H^r_{n,\lambda}(\alpha) t^n,$$

respectively.

The *generalized degenerate harmonic numbers of rank* r are defined by [5]

$$H_\lambda(n,r,\alpha) = \sum_{1 \leq k_0 + k_1 + \ldots + k_r \leq n} \frac{(-1)^{k_0+k_1+\cdots k_r + r+1}}{\lambda^{r+1} \alpha^{k_0+k_1+\ldots k_r}} \binom{\lambda}{k_0}\binom{\lambda}{k_1} \cdots \binom{\lambda}{k_r}$$

for $n > r \geq 0$. Note that $H_\lambda(n, r, \alpha) = 0$ for $n \leq r$ by convention. The generating function of these numbers is

$$\frac{\left(-\log_\lambda(1-\frac{t}{\alpha})\right)^{r+1}}{1-t} = \sum_{n=0}^{\infty} H_\lambda(n, r, \alpha)t^n. \tag{5}$$

When $\alpha = 1$, $H_\lambda(n, r, 1) = H_\lambda(n, r)$ are called the *degenerate harmonic numbers of rank* r.

The *degenerate Bernoulli polynomials* $\beta_{n,\lambda}(x)$ and the *degenerate Euler polynomials* $\varepsilon_{n,\lambda}(x)$ are defined by

$$\frac{te_\lambda^x(t)}{e_\lambda(t)-1} = \sum_{n=0}^{\infty} \beta_{n,\lambda}(x)\frac{t^n}{n!}, \tag{6}$$

and

$$\frac{2e_\lambda^x(t)}{e_\lambda(t)+1} = \sum_{n=0}^{\infty} \varepsilon_{n,\lambda}(x)\frac{t^n}{n!}, \tag{7}$$

respectively ([1], [2], and [14]). When $x = 0$, $\beta_{n,\lambda}(0) = \beta_{n,\lambda}$ are called the *degenerate Bernoulli numbers*, and $\varepsilon_{n,\lambda}(0) = \varepsilon_{n,\lambda}$ are called the *degenerate Euler numbers.*

The *degenerate Stirling numbers of the first kind* $S_{1,\lambda}(n, k)$ and the *degenerate Stirling numbers of the second kind* $S_{2,\lambda}(n, k)$ are defined by

$$(x)_n = \sum_{k=0}^{n} S_{1,\lambda}(n, k)(x)_{k,\lambda} \text{ for } n \geq 0,$$
$$(x)_{n,\lambda} = \sum_{k=0}^{n} S_{2,\lambda}(n, k)(x)_k \text{ for } n \geq 0,$$

respectively. The generating functions of these numbers are given by ([5], [8], [13], and [17])

$$\frac{(\log_\lambda(1+t))^k}{k!} = \sum_{n=k}^{\infty} S_{1,\lambda}(n, k)\frac{t^n}{n!} \quad \text{and} \quad \frac{(e_\lambda(t)-1)^k}{k!} = \sum_{n=k}^{\infty} S_{2,\lambda}(n, k)\frac{t^n}{n!}, \tag{8}$$

respectively, where k is a non-negative integer.

The *degenerate Daehee polynomials* are defined by [19]

$$\frac{\log_\lambda(1+t)}{t}(1+t)^x = \sum_{n=0}^{\infty} D_{n,\lambda}(x)\frac{t^n}{n!}. \tag{9}$$

When $x = 0$, $D_{n,\lambda}(0) = D_{n,\lambda}$ are called the *degenerate Daehee numbers.*

Recently, by using the concept of Riordan arrays, several identities pertaining to special numbers and binomial coefficients have been established ([23], [24]). Let

$g(t)$ and $f(t)$ be formal power series in the indeterminate, i.e., $g(t) = \sum_{n=0}^{\infty} g_n t^n$ and $f(t) = \sum_{n=0}^{\infty} f_n t^n$. A *Riordan array* is an infinite, lower triangular array and defined by a pair of functions $g(t)$ and $f(t)$ such that $R = (g(t), f(t)) = [r_{n,k}]_{n,k\geq 0}$ with

$$r_{n,k} = [t^n]\, g(t) \left(f(t)\right)^k, \tag{10}$$

where $g(0) \neq 0$, $f(0) = 0$, $f(1) \neq 0$, and $[t^n]$ defined by $[t^n]A(t) = a_n$ in the power series of function $A(t) = \sum_{n=0}^{\infty} a_n t^n$. An important example of Riordan arrays is the *Pascal triangle*, defined by $\left[\binom{n}{k}\right]_{n,k\geq 0} = \left(\frac{1}{1-t}, \frac{t}{1-t}\right)$, with the matrix

$$\left[\binom{n}{k}\right]_{n,k\geq 0} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 0 & 0 & 0 & \cdots \\ 1 & 2 & 1 & 0 & 0 & \cdots \\ 1 & 3 & 3 & 1 & 0 & \cdots \\ 1 & 4 & 6 & 4 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}. \tag{11}$$

Let $\mathcal{R}$ denote the set of Riordan arrays. It is known that $\langle \mathcal{R}, * \rangle$ forms a group under matrix multiplication $*$ with the identity $I = (1, 1)$ [22]:

$$(g(t), f(t)) * (h(t), l(t)) = (g(t)\, h(f(t)), l(f(t))). \tag{12}$$

Basically, the concept of Riordan arrays is used in a constructive way to find the generating function of many combinatorial sums. The summation property for a Riordan array $R = (g(t), f(t)) = [r_{n,k}]_{n,k\geq 0}$ is

$$\sum_{k=0}^{n} r_{n,k} h_k = [t^n]\, g(t)\, h(f(t)), \tag{13}$$

where $h(t) = \sum_{n=0}^{\infty} h_n t^n$.

2. Some Identities Using Riordan Arrays

In this section, we will give some identities using properties of Riordan arrays and generating functions. Some Riordan arrays for families of generalized degenerate harmonic numbers can be given as follows:

$$\left[H_{n-k+1,\lambda}(\alpha)\right]_{n,k} = \left(\frac{-\log_\lambda\left(1 - \frac{t}{\alpha}\right)}{(1-t)t}, t\right), \tag{14}$$

$$\left[H_{n-k+1,\lambda}^{k+1}(\alpha)\right]_{n,k} = \left(\frac{-\log_\lambda\left(1 - \frac{t}{\alpha}\right)}{(1-t)t}, \frac{t}{1-t}\right), \tag{15}$$

and

$$[H_\lambda(n+1,k,\alpha)]_{n,k} = \left(\frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{(1-t)t}, -\log_\lambda\left(1-\frac{t}{\alpha}\right) \right). \tag{16}$$

Theorem 1. *For a non-negative integer n, we have*

$$\sum_{k=0}^{n} (-1)^k H_{n-k+1,\lambda}^{k+1}(\alpha) = \frac{(-1)^n D_{n,\lambda}}{\alpha^{n+1} n!},$$

and for a non-negative integer r,

$$\sum_{k=0}^{n} \binom{k+r}{r} H_{n-k+1,\lambda}(\alpha) = H_{n+1,\lambda}^{r+2}(\alpha).$$

Proof. For the first identity, taking the Riordan array (15) and $h(t) = \frac{1}{1+t}$ in Equation (13), by Equations (5) and (9), we have

$$\begin{aligned}\sum_{k=0}^{n} (-1)^k H_{n-k+1,\lambda}^{k+1}(\alpha) =& [t^n] \frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{(1-t)t} \frac{1}{1+\frac{t}{1-t}} = [t^n] \frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{t} \\ =& [t^n] \sum_{n=0}^{\infty} \frac{(-1)^n D_{n,\lambda}}{\alpha^{n+1} n!} t^n = \frac{(-1)^n D_{n,\lambda}}{\alpha^{n+1} n!}.\end{aligned}$$

So, we have the first identity. Similarly, using the Riordan array (14) and taking $h(t) = \frac{1}{(1-t)^{r+1}}$ in Equation (13), by Equation (4), the desired result is obtained. □

Theorem 2. *For a positive integer n, we have*

$$\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} H_{k,\lambda} = \frac{1}{\lambda} \binom{n+\lambda-1}{n}.$$

Proof. By choosing Riordan array $R = \left(\frac{-1}{1-t}, \frac{-t}{1-t}\right) = \left[(-1)^{k+1}\binom{n}{k}\right]_{n,k}$ and $h(t) = \frac{-\log_\lambda(1-t)}{1-t}$ in Equation (13), using Equations (2) and (3), we have

$$\begin{aligned}\sum_{k=0}^{n} (-1)^{k+1} \binom{n}{k} H_{k,\lambda} &= [t^n] \frac{-1}{1-t} \frac{-\log_\lambda\left(1+\frac{t}{1-t}\right)}{1+\frac{t}{1-t}} = [t^n] \frac{\left(\frac{1}{1-t}\right)^\lambda - 1}{\lambda} \\ &= \frac{1}{\lambda} [t^n] \left(\sum_{n=0}^{\infty} \binom{n+\lambda-1}{n} t^n - 1 \right) \\ &= \frac{1}{\lambda} [t^n] \sum_{n=1}^{\infty} \binom{n+\lambda-1}{n} t^n \\ &= \frac{1}{\lambda} \binom{n+\lambda-1}{n}\end{aligned}$$

for $n \geq 1$. So, the proof is complete. □

Theorem 3. *For a non-negative integer n, we have*

$$\sum_{k=0}^{n} \frac{(-1)^k (1)_{k,\lambda}}{k!} H_\lambda(n+1,k,\alpha) = H_{n+1,\lambda}(\alpha) - \frac{H_{n,\lambda}(\alpha)}{\alpha}.$$

Proof. Taking the Riordan array (16) and $h(t) = e_\lambda(-t)$ in Equation (13), by Equations (1) and (5), we have

$$\begin{aligned}\sum_{k=0}^{n} H_\lambda(n+1,k,\alpha)\frac{(-1)^k (1)_{k,\lambda}}{k!} &= [t^n]\frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{(1-t)t}\left(1-\frac{t}{\alpha}\right)\\ &= [t^{n+1}]\frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{1-t} - \frac{1}{\alpha}[t^n]\frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{1-t}\\ &= H_{n+1,\lambda}(\alpha) - \frac{H_{n,\lambda}(\alpha)}{\alpha}\end{aligned}$$

for $n \geq 0$. So, the proof is complete. □

Theorem 4. *For non-negative integers n and r, we have*

$$\sum_{k=0}^{n} H_\lambda(k,r+1,\alpha) = \sum_{k=0}^{n} H_{n-k,\lambda}(\alpha) H_\lambda(k,r,\alpha).$$

Proof. Taking the Riordan array $[r_{n,k}]_{n,k\geq 0} = \left(\frac{1}{1-t}, t\right)$, which means $r_{n,k} = 1$, when $n \geq k$ and $h(t) = \frac{\left(-\log_\lambda(1-\frac{t}{\alpha})\right)^{r+2}}{1-t}$ in Equation (13), we have

$$\begin{aligned}\sum_{k=0}^{n} H_\lambda(k,r+1,\alpha) &= [t^n]\frac{-\log_\lambda(1-\frac{t}{\alpha})}{1-t}\frac{\left(-\log_\lambda(1-\frac{t}{\alpha})\right)^{r+1}}{1-t}\\ &= [t^n]\sum_{n=0}^{\infty} H_{n,\lambda}(\alpha)t^n \sum_{n=0}^{\infty} H_\lambda(n,r,\alpha)t^n\\ &= \sum_{k=0}^{n} H_{n-k,\lambda}(\alpha)H_\lambda(k,r,\alpha),\end{aligned}$$

as claimed. □

Theorem 5. *For a non-negative integer n, we have*

$$\sum_{k=0}^{n} \frac{(-1)^k (1)_{k+1,\lambda}}{(k+1)!} H_\lambda(n+1,k,\alpha) = \frac{1}{\alpha}.$$

Proof. Taking the Riordan array (16) and

$$h(t) = \frac{1-e_\lambda(-t)}{t} = \sum_{n=0}^{\infty} \frac{(-1)^n (1)_{n+1,\lambda} t^n}{(n+1)!}$$

in Equation (13), we have

$$\begin{aligned}\sum_{k=0}^{n}\frac{(-1)^k(1)_{k+1,\lambda}}{(k+1)!}H_\lambda(n+1,k,\alpha) &= [t^n]\frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{(1-t)t}h\left(-\log_\lambda\left(1-\frac{t}{\alpha}\right)\right)\\ &= [t^n]\frac{1}{1-t}\frac{1}{\alpha}\\ &= \frac{1}{\alpha}.\end{aligned}$$

So, the proof is complete. □

Theorem 6. *For non-negative integers n and r, we have*

$$\sum_{k=r}^{n}\frac{(-1)^{n-k}}{\lambda\alpha^{n-k+1}}\binom{\lambda}{n-k+1}\binom{k}{r} = H_{n-r+1,\lambda}^{r+1}(\alpha).$$

Proof. By Equation (12), we write

$$\left(\frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{(1-t)t},\frac{t}{1-t}\right) = \left(\frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{t},t\right) * \left(\frac{1}{1-t},\frac{t}{1-t}\right). \tag{17}$$

By Equations (2) and (10), we have

$$\left[\frac{(-1)^{n-k}}{\lambda\alpha^{n-k+1}}\binom{\lambda}{n-k+1}\right]_{n,k} = \left(\frac{-\log_\lambda\left(1-\frac{t}{\alpha}\right)}{t},t\right),$$

and from Matrix (11) and Equation (17), we write

$$\left[H_{n-k+1,\lambda}^{k+1}(\alpha)\right]_{n,k} = \left[\frac{(-1)^{n-k}}{\lambda\alpha^{n-k+1}}\binom{\lambda}{n-k+1}\right]_{n,k}\left[\binom{n}{k}\right]_{n,k},$$

which completes the proof by using matrix multiplication. □

3. Some Identities Involving $H_\lambda(n,r,\alpha)$

Let θ_n be any sequence, and its generating function be $\Theta(t)=\sum_{n=0}^{\infty}\theta_n t^n$. Since $H_\lambda(n,r,\alpha)=0$ when $n\le r$, we can write

$$\begin{aligned}\sum_{r=1}^{n}\theta_r H_\lambda(n,r-1,\alpha) = \sum_{r=1}^{\infty}\theta_r H_\lambda(n,r-1,\alpha) &= \sum_{r=1}^{\infty}\theta_r[t^n]\frac{\left(-\log_\lambda(1-\frac{t}{\alpha})\right)^r}{1-t}\\ &= [t^n]\frac{\Theta\left(-\log_\lambda(1-\frac{t}{\alpha})\right)-\theta_0}{1-t}.\end{aligned} \tag{18}$$

According to this scheme, we give some identities involving $H_\lambda(n,r,\alpha)$ with the following theorems.

Theorem 7. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n}\frac{(1)_{k,\lambda}}{k!}H_\lambda(n,k-1,\alpha)=\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{(-1)^{k+i}(1)_{i,\lambda}S_{1,\lambda}(k,i)}{\alpha^k k!}.$$

Proof. Let $\theta_k=\frac{(1)_{k,\lambda}}{k!}$, that is, $\Theta(t)=e_\lambda(t)$ in Equation (18). Using the binomial theorem, Equations (1) and (8), we have

$$\begin{aligned}
e_\lambda\left(-\log_\lambda(1-\tfrac{t}{\alpha})\right)-\theta_0&=\left(1-\lambda\log_\lambda(1-\tfrac{t}{\alpha})\right)^{1/\lambda}-1\\
&=\sum_{i=1}^{\infty}(-1)^i\binom{1/\lambda}{i}\lambda^i(\log_\lambda(1-\tfrac{t}{\alpha}))^i\\
&=\sum_{i=1}^{\infty}(-1)^i(1)_{i,\lambda}\sum_{n=i}^{\infty}\frac{(-1)^nS_{1,\lambda}(n,i)}{\alpha^n n!}t^n\\
&=\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{(-1)^{n+i}(1)_{i,\lambda}S_{1,\lambda}(n,i)}{\alpha^n n!}t^n.
\end{aligned}$$

Then by Equation (18), we get

$$\begin{aligned}
\sum_{k=0}^{n}\frac{(1)_{k,\lambda}}{k!}H_\lambda(n,k-1,\alpha)&=[t^n]\frac{1}{1-t}e_\lambda\left(-\log_\lambda(1-\tfrac{t}{\alpha})-1\right)\\
&=[t^n]\left(\sum_{n=0}^{\infty}t^n\right)\left(\sum_{n=1}^{\infty}\sum_{i=1}^{n}\frac{(-1)^{n+i}(1)_{i,\lambda}S_{1,\lambda}(n,i)}{\alpha^n n!}t^n\right)\\
&=\sum_{k=1}^{n}\sum_{i=1}^{k}\frac{(-1)^{k+i}(1)_{i,\lambda}S_{1,\lambda}(k,i)}{\alpha^k k!}.
\end{aligned}$$

So, we have the proof. □

Theorem 8. *Let n be a positive integer. Then*

$$\sum_{k=1}^{n}\frac{(-1)^k\beta_{k,\lambda}H_\lambda(n,k-1,\alpha)}{k!}=\alpha H_{n+1,\lambda}(\alpha)-1,$$

and

$$\sum_{k=1}^{n}\frac{(-1)^{k+1}\varepsilon_{k,\lambda}}{k!}H_\lambda(n,k-1,\alpha)=\frac{(2\alpha)^{-n}-1}{2\alpha-1},\quad(\alpha\neq 1/2).$$

Proof. For the first identity, let $\Theta(t)=\frac{-t}{e_\lambda(-t)-1}=\sum_{k=0}^{\infty}\frac{(-1)^k\beta_{k,\lambda}}{k!}t^k$ in Equation

(18). By Equation (6), we have

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^{k}\,\beta_{k,\lambda}H_{\lambda}(n,k-1,\alpha)}{k!} &= [t^{n}]\frac{1}{1-t}\left(\frac{\log_{\lambda}(1-\frac{t}{\alpha})}{e_{\lambda}\left(\log_{\lambda}(1-\frac{t}{\alpha})\right)-1}-1\right)\\ &= [t^{n}]\frac{\log_{\lambda}(1-\frac{t}{\alpha})}{1-t}\frac{-\alpha}{t}-[t^{n}]\frac{1}{1-t}\\ &= \alpha H_{n+1,\lambda}(\alpha)-1.\end{aligned}$$

For the second identity, let $\Theta(t)=\frac{2}{e_{\lambda}(-t)+1}=\sum_{k=0}^{\infty}\frac{(-1)^{k}\varepsilon_{k,\lambda}}{k!}t^{k}$ in Equation (18). In a similar way, by Equation (7), for $\alpha\neq 1/2$, we get

$$\begin{aligned}\sum_{k=1}^{n}\frac{(-1)^{k+1}\varepsilon_{k,\lambda}H_{\lambda}(n,k-1,\alpha)}{k!} &= -[t^{n}]\frac{1}{1-t}\left(\frac{2}{2-\frac{t}{\alpha}}-1\right)\\ &= -[t^{n}]\sum_{n=0}^{\infty}t^{n}\sum_{n=1}^{\infty}\frac{1}{2^{n}\alpha^{n}}t^{n}\\ &= -\sum_{k=1}^{n}\frac{1}{2^{k}\alpha^{k}}\\ &= \frac{(2\alpha)^{-n}-1}{2\alpha-1},\end{aligned}$$

as claimed. So, the proof is complete. □

Theorem 9. *Let n and m be positive integers such that $n\geq m$. Then we have*

$$\sum_{k=m}^{n}\frac{(-1)^{k}S_{2,\lambda}(k,m)H_{\lambda}(n,k-1,\alpha)}{k!}=\frac{(-1)^{m}}{m!\alpha^{m}}.$$

Proof. Let $\Theta(t)=\frac{(e_{\lambda}(-t)-1)^{m}}{m!}$. By the fact that $S_{2,\lambda}(n,m)=0$ when $n<m$ and Equation (18), we have

$$\begin{aligned}\sum_{k=m}^{n}\frac{(-1)^{k}S_{2,\lambda}(k,m)H_{\lambda}(n,k-1,\alpha)}{k!} &= [t^{n}]\frac{1}{1-t}\frac{(-\frac{t}{\alpha})^{m}}{m!}\\ &= \frac{(-1)^{m}}{m!\alpha^{m}}[t^{n-m}]\frac{1}{1-t}\\ &= \frac{(-1)^{m}}{m!\alpha^{m}}.\end{aligned}$$

Hence, the proof is complete. □

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