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MAKER-BREAKER RADO GAMES FOR EQUATIONS WITH RADICALS

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Abstract

We study two-player positional games where Maker and Breaker take turns to select a previously unoccupied number in $\{1, 2, \ldots, n\}$. Maker wins if the numbers selected by Maker contain a solution to the equation

$$x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$$

where k and ℓ are integers with $k \geq 2$ and $\ell \neq 0$, and Breaker wins if they can stop Maker. Let $f(k, \ell)$ be the smallest positive integer n such that Maker has a winning strategy when $x_1, \ldots, x_k$ are not necessarily distinct, and let $f^*(k, \ell)$ be the smallest positive integer n such that Maker has a winning strategy when $x_1, \ldots, x_k$ are distinct. When $\ell \geq 1$, we prove that, for all $k \geq 2$, $f(k, \ell) = (k + 2)^\ell$ and $f^*(k, \ell) = (k^2 + 3)^\ell$; when $\ell \leq -1$, we prove that $f(k, \ell) = [k + \Theta_k(1)]^{-\ell}$ and $f^*(k, \ell) = [\exp(O_k(k \log k))]^{-\ell}$. Our proofs use elementary combinatorial arguments as well as results from number theory and arithmetic Ramsey theory.

1. Introduction

Let $\mathcal{F}$ be a family of finite subsets of $\mathbb{N} := \{1, 2, \ldots\}$ and $n \in \mathbb{N}$. Maker-Breaker games played on $[n] := \{1, 2, \ldots, n\}$ with winning sets $\mathcal{F}$ are two-player positional games where Maker and Breaker take turns to select a previously unoccupied number in $[n]$. Maker goes first. Maker wins if they can occupy a set in $\mathcal{F}$ and Breaker wins otherwise. The van der Waerden games introduced by Beck [1] are games of this type. In van der Waerden games, $\mathcal{F}$ is the set of k-term arithmetic progressions for a fixed k. These games were motivated by a result of van der Waerden's

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theorem [24] which says that if $\mathbb{N}$ is partitioned into two classes, then one of them contains arbitrarily long arithmetic progressions. By the compactness principle [10, Chapter 1] (see also [18, Section 2.1]) and strategy stealing [2, Section 5] (see also [14, Chapter 1]), Maker can win the van der Waerden games if n is large enough. Therefore, one would naturally want to find the smallest n such that Maker can win the van der Waerden games. Beck [1] proved that, for any given k, the smallest n such that Maker has a winning strategy for the van der Waerden games is between $2^{k-7k^{7/8}}$ and $k^3 2^{k-4}$.

Recently, Kusch, Rué, Spiegel, and Szabó [17] studied a generalization of van der Waerden games called Rado games. In Rado games, $\mathcal{F}$ is the set of solutions to a system of linear equations. By Rado's theorem [22], if n is large enough, then Maker is guaranteed to win the Rado games if the system of linear equations satisfies the so-called column condition [10, Chapter 10]. Kusch, Rué, Spiegel, and Szabó allowed Breaker to select $q \geq 1$ numbers each round and derived asymptotic thresholds of q for Breaker to win. Their result on 3-term arithmetic progressions was later improved by Cao et al. [7]. Hancock [12] replaced $[n]$ with a random subset of $[n]$ where each number is included with probability p and proved asymptotic thresholds of p for Breaker or/and Maker to win. However, unlike the van der Waerden games, the smallest n such that Maker wins for the unbiased and deterministic Rado games are left unstudied.

In this paper, we study the smallest positive integer n such that Maker wins the Rado games on $[n]$ when $\mathcal{F}$ is the set of solutions to the equation

$$x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell} \tag{1}$$

where k and ℓ are integers with $k \geq 2$ and $\ell \neq 0$. Equation (1) is connected with results in arithmetic Ramsey theory [10, 18]. In arithmetic Ramsey theory, a system of equations $E(x_1, \ldots, x_k, y) = 0$ in variables $x_1, \ldots, x_k, y$ is called *partition regular* if whenever $\mathbb{N}$ is partitioned into a finite number of classes, one of them contains a solution to $E(x_1, \ldots, x_k, y) = 0$. In 1991, Lefmann [19] proved that, among other things, Equation (1) is partition regular for all $\ell \in \mathbb{Z}\backslash\{0\}$. In the same year, Brown and Rödl [6] proved that if a system $E(x_1, \ldots, x_k, y) = 0$ of homogeneous equations is partition regular, then the system $E(1/x_1, \ldots, 1/x_k, 1/y) = 0$ is also partition regular.

To state our results, we first define the games we study in detail. Let $A \subseteq \mathbb{N}$ be a finite set and let $e(x_1, \ldots, x_k, y) = 0$ be an equation in variables $x_1, \ldots, x_k, y$. The Maker-Breaker Rado games denoted

$$G(A, e(x_1, \ldots, x_k, y) = 0) \text{ and } G^*(A, e(x_1, \ldots, x_k, y) = 0)$$

have the following rules:

(1) Maker and Breaker take turns to select a number from A. Once a number

is selected by a player, neither players can select that number again. Maker starts the game.

(2) Maker wins the $G(A, e(x_1, \ldots, x_k, y) = 0)$ game if a collection of the numbers chosen by Maker form a solution to $e(x_1, \ldots, x_k, y) = 0$ where $x_1, \ldots, x_k$ are *not* necessarily distinct; and Maker wins the $G^*(A, e(x_1, \ldots, x_k, y) = 0)$ game if a collection of the numbers chosen by Maker form a solution to $e(x_1, \ldots, x_k, y) = 0$ where $x_1, \ldots, x_k$ are distinct.

(3) Breaker wins if Maker fails to occupy a solution to $e(x_1, \ldots, x_k, y) = 0$.

We use the following shorter notations for games with Equation (1):

$$G([n], k, \ell) := G\left([n], x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}\right)$$

and

$$G^*([n], k, \ell) := G^*\left([n], x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}\right).$$

We say that a player wins a game if there is a winning strategy which guarantees that this player wins no matter what the other player does. A *winning strategy* is a set of instructions which tells the player what to do each round given what had been previously played by both players. Let $f(k, \ell)$ be the smallest positive integer n such that Maker wins the $G([n], k, \ell)$ game and let $f^*(k, \ell)$ be the smallest positive integer n such that Maker wins the $G^*([n], k, \ell)$ game.

For $\ell \geq 1$, we are able to find exact formulas for $f(k, \ell)$ and $f^*(k, \ell)$.

Theorem 1. *For all integers $k \geq 2$ and $\ell \geq 1$, we have $f(k, \ell) = (k+2)^\ell$.*

Theorem 2. *For all integers $k \geq 2$ and $\ell \geq 1$, we have $f^*(k, \ell) = (k^2+3)^\ell$.*

Our proofs of Theorems 1 and 2 involve showing that $f(k, 1) = k+2$ and $f^*(k, 1) = k^2 + 3$ using elementary combinatorial arguments, and that $f(k, \ell) \leq [f(k, 1)]^\ell$ and $f^*(k, \ell) \leq [f^*(k, 1)]^\ell$ using a result of Besicovitch [3] on the linear independence of integers with fractional powers.

For $\ell \leq -1$, our main results are the following:

Theorem 3. *Let k, ℓ be integers with $k \geq 2$ and $\ell \leq -1$. Then $f(k, \ell) = [k + \Theta_k(1)]^{-\ell}$. More specifically, if $k \geq 1/(2^{-1/\ell} - 1)$, then $f(k, \ell) \geq (k+1)^{-\ell}$; and if $k \geq 4$, then $f(k, \ell) \leq (k+2)^{-\ell}$.*

Theorem 4. *Let k, ℓ be integers with $k \geq 2$ and $\ell \leq -1$. Then $f^*(k, \ell) = [\exp(O_k(k \log k))]^{-\ell}$.*

The proof of Theorem 4 involves showing that $f^*(k, -1) = \exp(O_k(k \log k))$ using a game theoretic variant of a theorem in arithmetic Ramsey theory by Brown and Rödl [6].

Our results indicate that it is "easier" to form a solution to Equation (1) strategically compared to their counterparts in arithmetic Ramsey theory. To illustrate this, let $R(k, \ell)$ be the smallest positive integer n such that if $[n]$ is partitioned into two classes then one of them has a solution to Equation (1) with $x_1, \dots, x_k$ not necessarily distinct, and let $R^*(k, \ell)$ be the smallest positive integer n such that if $[n]$ is partitioned into two classes then one of them has a solution to Equation (1) with $x_1, \dots, x_k$ distinct. Note that if Maker and Breaker choose numbers in $[n]$, with $n \geq R(k, \ell)$ (respectively, $n \geq R^*(k, \ell)$), until there is no number left to choose, then the sets of numbers chosen by Maker and Breaker form a partition of $[n]$. If Maker does not win the game, then it means that the set of numbers chosen by Breaker contains a solution to Equation (1). Since Maker goes first, by strategy stealing, Maker could follow Breaker's strategy and win the game. Therefore, we have $f(k, \ell) \leq R(k, \ell)$ and $f^*(k, \ell) \leq R^*(k, \ell)$. When $\ell \in \{-1, 1\}$, some results on $R(k, \ell)$ and $R^*(k, \ell)$ are known.

For $\ell = 1$, Beutelsapacher and Brestovansky [4] proved that $R(k, 1) = k^2 + k - 1$. The exact formula for $R^*(k, 1)$ is not known, but Boza, Revuelta, and Sanz [5] proved that, for $k \geq 6$, $R^*(k, 1) \geq (k^3 + 3k^2 - 2k)/2$. Hence, by Theorems 1 and 2, we have

$$\lim_{k\to\infty} \frac{f(k,1)}{R(k,1)} = \lim_{k\to\infty} \frac{f^*(k,1)}{R^*(k,1)} = 0.$$

For $\ell = -1$, Myers and Parrish [20] calculated that $R(2, -1) = 60$, $R(3, -1) = 40$, $R(4, -1) = 48$, and $R(5, -1) = 39$; and the first author [9] proved that $R(k, -1) \geq k^2$. So by Theorem 3, we have

$$\lim_{k\to\infty} \frac{f(k,-1)}{R(k,-1)} = 0. \tag{2}$$

Unfortunately, we do not know a similar lower bound for $R^*(k, -1)$. However, we believe that Maker can still do better by selecting numbers strategically.

Conjecture 1. $\lim_{k\to\infty} f^*(k, -1)/R^*(k, -1) = 0$.

This paper is organized as follows. We first prove some preliminary results in Section 2. The next four sections are devoted to proving Theorems 1 to 4. In Section 7, we study Rado games for linear equations with arbitrary coefficients. We discuss some future research directions in Section 8.

1.1. Asymptotic Notation

We use standard asymptotic notation. For functions $f(k)$ and $g(k)$, $f(k) = O_k(g(k))$ if there exist constants K and C such that $|f(k)| \leq C|g(k)|$ for all $k \geq K$; $f(k) = \Omega_k(g(k))$ if there exist constants K' and c such that $|f(k)| \geq c|g(k)|$ for all $k \geq K'$; $f(k) = \Theta_k(g(k))$ if $f(k) = O_k(g(k))$ and $f(k) = \Omega_k(g(k))$; and $f(k) = o_k(g(k))$ if $\lim_{k\to\infty} f(k)/g(k) = 0$.

We remind the reader that, throughout this paper, *we only use asymptotic notation for functions of* k where ℓ is neither a parameter nor a constant.

2. Preliminaries

We prove some results which will be used to prove Theorems 1 to 4. Our first result shows that the games for equations with radicals can be partially reduced to games for equation without radicals, i.e., $\ell = 1$ or $\ell = -1$.

Lemma 1. *Let* k *and* ℓ *be integers with* $k \geq 2$ *and* $\ell \neq 0$. *If* $\ell \geq 1$, *then*

$$f(k, \ell) \leq [f(k,1)]^{\ell} \text{ and } f^*(k,\ell) \leq [f(k,1)]^{\ell}.$$

If $\ell \leq -1$, *then*

$$f(k, \ell) \leq [f(k,-1)]^{-\ell} \text{ and } f^*(k,\ell) \leq [f(k,-1)]^{-\ell}.$$

Proof. We prove that if $\ell \geq 1$, then $f(k, \ell) \leq [f(k,1)]^{\ell}$. The other inequalities can be proved similarly.

Write $M = f(k, 1)$ and let $\mathcal{M}$ be a Maker's winning strategy for the $G([M], k, 1)$ game. Notice that if $(x_1, \ldots, x_k, y) = (a_1, \ldots, a_k, b)$ is a solution to $x_1 + \cdots + x_k = y$, then $(x_1, \ldots, x_k, y) = (a_1^{\ell}, \ldots, a_k^{\ell}, b^{\ell})$ is a solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$.

For $i = 1, 2, \ldots$, let $m_i \in [M^{\ell}]$ be the number chosen by Maker and let $b_i \in [M^{\ell}]$ be the number chosen by Breaker in round i. We define a strategy for Maker recursively. We note that Maker focuses on the set $\{1^{\ell}, 2^{\ell}, \ldots, M^{\ell}\}$ in this strategy. In round 1, if $\mathcal{M}$ tells Maker to choose a_1 for the $G([M], k, 1)$ game, then set $m_1 = a_1^{\ell}$. If $b_1 = z_1^{\ell}$ for some $z_1 \in [M]$, then set $b_1' = z_1$; otherwise, arbitrarily set b_1' equal to some number in $M \backslash \{a_1\}$. In round $i \geq 2$, given $a_1, a_2, \ldots, a_{i-1}, b_1', b_2', \ldots, b_{i-1}'$, if $\mathcal{M}$ tells Maker to choose a_i, then set $m_i = a_i$. This is possible because $\mathcal{M}$ is a winning strategy. If $b_i = z_i^{\ell}$ for some $z_i \in [M]$, then set $b_i' = z_i$; otherwise, arbitrarily set b_i' equal to some number in $M \backslash \{a_1, a_2, \ldots, a_{i-1}, a_i, b_1', b_2', \ldots, b_{i-1}'\}$.

Now since $\mathcal{M}$ is a winning strategy, there exists t such that $\{a_1, a_2, \ldots, a_t\}$ has a solution to $x_1 + \cdots + x_k = y$. Hence $\{m_1, m_2, \ldots, m_t\} = \{a_1^{\ell}, a_2^{\ell}, \ldots, a_t^{\ell}\}$ has a solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$. Therefore, Maker wins the $G([M^{\ell}], k, \ell)$ game. □

Theorems 1 and 2 indicate that these inequalities in Lemma 1 are actually equalities when $\ell \geq 2$. This is due to a result of Besicovitch [3]. To state this result, we first need the following definition.

Definition 1. Let $a \in \mathbb{N} \backslash \{1\}$. We say that a is *power-ℓ free* if $a = b^{\ell} c$, with $b, c \in \mathbb{N}$, implies $b = 1$.

Theorem 5 (Besicovitch [3]). *For all positive integers $\ell \geq 2$, the set*

$$A(\ell) := \{a^{1/\ell} : a \in \mathbb{N}\backslash\{1\} \text{ and } a \text{ is power-}\ell \text{ free}\}$$

is linearly independent over $\mathbb{Z}$. That is, if $a_1, \ldots, a_m \in A(\ell)$ and $c_1, \ldots, c_m \in \mathbb{N}$ satisfy $c_1a_1 + \cdots + c_ma_m = 0$, then $c_1 = \cdots = c_m = 0$.

Besicovitch [3] actually provided an elementary proof of a stronger result, but Theorem 5 is enough for our purposes. For interested readers, we note that Richards [23] proved a similar result to the one in [3], but using Galois theory instead. A direct consequence of Theorem 5 is the following result which will be used in proving Theorems 1 and 2.

Corollary 1. *Let k, ℓ be integers with $k \geq 2$ and $\ell \geq 1$. The solutions to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$ are of the form $(x_1, \ldots, x_k, y) = (ca_1^\ell, \ldots, ca_k^\ell, cb^\ell)$ where $a_1, \ldots, a_k, b, c \in \mathbb{N}$, $a_1 + \cdots + a_k = b$, and c is power-ℓ free.*

Proof. Suppose that $\alpha_1, \ldots, \alpha_k, \beta \in \mathbb{N}$ satisfy

$$\alpha_1^{1/\ell} + \cdots + \alpha_k^{1/\ell} = \beta^{1/\ell}.$$

We write $\alpha_i = c_ia_i^\ell$ for all $i = 1, ..., k$, and $\beta = db^\ell$ where $a_1, \ldots, a_k, c_1, \ldots, c_k, b, d \in \mathbb{N}$ and $c_1, \ldots, c_k, d$ are power-ℓ free. Then we have

$$a_1c_1^{1/\ell} + \cdots + a_kc_k^{1/\ell} - bd^{1/\ell} = 0. \tag{3}$$

We first show that $c_1 = \cdots = c_k = d$. Suppose, for a contradiction, that $c_1, \ldots, c_k, d$ are not all the same. We split this into two cases.

`Case 1:` $d \neq c_i$ for all $i \in [k]$. After combining terms with the same ℓ-th roots, the left-hand side of Equation (3) has at least two terms where one of them is $-bd^{1/\ell}$. Now by Theorem 5, $b = 0$ which is a contradiction.

`Case 2:` $d = c_i$ for some $i \in [k]$. Then there exists $j \in [k]\backslash\{i\}$ such that $c_j \neq c_i$. After combining terms with the same ℓ-th roots, the left-hand side of Equation (3) has a term with $c_j^{1/\ell}$. This is because all the terms with $c_j^{1/\ell}$ contain only positive coefficients. By Theorem 5, the coefficient of $c_j^{1/\ell}$ is zero after combining like terms. But this is impossible because the coefficient of $c_j^{1/\ell}$ is the sum of a subset of $\{a_1, ..., a_k\}$ consisting only positive integers.

Hence we have $c_1 = \cdots = c_k = d$. Therefore, $a_1 + \cdots + a_k = b$. □

We note that Newman [21] proved Corollary 1 for the case $k = 2$ without using Theorem 5.

Next, we prove a game theoretic variant of a result by Brown and Rödl [6, Theorem 2.1]. We note that an equation $e(x_1, \ldots, x_k, y) = 0$ is *homogeneous* if whenever $(x_1,\ldots,x_k,y)=(a_1,\ldots,a_k,b)$ is a solution to $e(x_1,\ldots,x_k,y)=0$, for all $m\in\mathbb{N}$, $(x_1,\ldots,x_k,y)=(ma_1,\ldots,ma_k,mb)$ is a also a solution to $e(x_1,\ldots,x_k,y)=0$.

Theorem 6. *Let A be a finite subset of $\mathbb{N}$, L the least common multiple of A, $k\in\mathbb{N}$, and $e(x_1,\ldots,x_k,y)=0$ a homogeneous equation. If Maker wins the $G(A,e(x_1,\ldots,x_k,y)=0)$ game, then Maker wins the $G([L],e(1/x_1,\ldots,1/x_k,1/y)=0)$ game. Similarly, if Maker wins the $G^*(A,e(x_1,\ldots,x_k,y)=0)$ game, then Maker wins the $G^*([L],e(1/x_1,\ldots,1/x_k,1/y)=0)$ game.*

Proof. Suppose that Maker wins the $G(A,e(x_1,\ldots,x_k,y)=0)$ game. Let $\mathcal{M}$ be a Maker's winning strategy. We consider the following Maker's strategy for the $G([L],e(1/x_1,\ldots,x_k,1/y)=0)$ game. In round 1, if $\mathcal{M}$ tells Maker to choose m_1 for the $G(A,e(x_1,\ldots,x_k,y)=0)$ game, then Maker chooses $L/m_1\in\{1,\ldots,L\}$. The rest of the strategy is defined inductively. For all rounds i, let L/b_i be the number chosen by Breaker and L/m_i be the number chosen by Maker where $m_i\in\{1,\ldots,L\}$. If $b_i\in A$, then we set $b_i'=b_i$; if $b_i\notin A$, then arbitrarily set b_i' equal to some number in $A\backslash\{m_1,\ldots,m_i,b_1',\ldots,b_{i-1}'\}$. For all rounds $i\geq 2$, given $\{m_1,\ldots,m_{i-1},b_1',\ldots,b_{i-1}'\}$, if $\mathcal{M}$ tells Maker to choose m_i for the $G(A,e(x_1,\ldots,x_k,y)=0)$ game, then Maker chooses L/m_i for the

$$G([L],e(1/x_1,\ldots,1/x_k,1/y)=0)$$

game. This process is possible because $\mathcal{M}$ is a winning strategy.

Since $\mathcal{M}$ is a winning strategy, in some round t, there exists a subset $\{a_1,\ldots,a_s\}$ of $\{m_1,\ldots,m_t\}$ which form a solution to $e(x_1,\ldots,x_k,y)=0$. By homogeneity, $\{L/a_1,\ldots,L/a_s\}$ form a solution to $e(1/x_1,\ldots,1/x_k,1/y)=0$. So Maker wins the $G([L],e(1/x_1,\ldots,1/x_k,1/y)=0)$ game.

The case for the $G^*([L],e(1/x_1,\ldots,1/x_k,1/y)=0)$ game can be proved in a similar way. $\square$

The key feature of Theorem 6 is that one can choose a set A whose least common multiple L is small. This was not used by Brown and Rödl [6, Theorem 2.1]. For interested readers, we note that the first author [9] recently improved a quantitative result by Brown and Rödl [6, Theorem 2.5] with the help of this observation.

Finally, we also need the following definitions.

Definition 2. Given $m\in\mathbb{N}$ mutually disjoint subsets $\{s_1,t_1\}$, $\{s_2,t_2\}$, $\ldots$, $\{s_m,t_m\}$ of $\mathbb{N}$ with size 2, the *pairing strategy* over those disjoint subsets for a player is defined as follows: if their opponent chooses s_i for some $i=1,2,\ldots,m$, then this player chooses t_i.

Definition 3. Let $k\geq 2$ be an integer and $a_1x_1+\cdots+a_kx_k=y$ a linear equation. Suppose, at some point of the $G^*([n],a_1x_1+\cdots+a_kx_k=y)$ game, Maker has claimed a set A of at least k integers. Then we call $a_1\alpha_1 + \cdots + a_k\alpha_k$ a *k-sum* for any k distinct integers $\alpha_1, \ldots, \alpha_k \in A$.

3. Proof of Theorem 1

We first prove Theorem 1 for the case $\ell = 1$.

Lemma 2. *For all integers $k \geq 2$, we have $f(k, 1) = k + 2$.*

Proof. We first show that Maker wins the $G([k+2], k, 1)$ game. Note that this will be proved in more full generality later in Theorem 8. We consider two cases.
`Case 1:` $k = 2$. Maker starts by choosing 2. Since $2+2 = 4$ and $1+1 = 2$, Maker wins the game in the next round by choosing either 1 or 4, whichever is available.
`Case 2:` $k > 2$. Maker starts by selecting 1. Notice that

$$\underbrace{1 + 1 + \cdots + 1}_{k} = k \cdot 1 = k,$$

$$\underbrace{1 + 1 + \cdots + 1}_{k-1} + 2 = (k-1) \cdot 1 + 2 = k + 1,$$

and

$$\underbrace{1 + 1 + \cdots + 1}_{k-2} + 2 + 2 = (k-2) \cdot 1 + 2 \cdot 2 = k + 2.$$

If Breaker chooses k in the first round, then Maker chooses 2 in round 2 and wins the game in round 3 by choosing either $k+1$ or $k+2$. If Breaker does not choose k in round 1, then Maker can win the game in round 2 by choosing k.

Now we show that Breaker wins the $G([k+1], k, 1)$ game. When $\ell = 1$, the only possible solutions to Equation (1) in $\{1, \ldots, k+1\}$ are

$$(x_1, x_2, \ldots, x_{k-1}, x_k, y) = (1, 1, \ldots, 1, 1, k)$$

and

$$(x_1, x_2, \ldots, x_{k-1}, x_k, y) = (1, 1, \ldots, 1, 2, k+1).$$

If $k = 2$, then Breaker wins the game by the pairing strategy over $\{1, 2\}$. If $k \geq 3$, then Breaker wins the game by the pairing strategy over $\{1, k\}$ and $\{2, k+1\}$. $\square$

We also need a result on the solutions to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$ in $\{1, 2, \ldots, (k+2)^\ell - 1\}$ when k, ℓ are integers with $k \geq 2$ and $\ell \geq 1$.

Lemma 3. *For all integers $k \geq 2$ and $\ell \geq 1$, the only solutions to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$ in $\{1, 2, \ldots, (k+2)^\ell - 1\}$ are*

$$(x_1, \ldots, x_{k-2}, x_{k-1}, x_k, y) = (a, \ldots, a, a, a, ak^\ell),$$

and

$$(x_1, \ldots, x_{k-2}, x_{k-1}, x_k, y) = (b, \ldots, b, b, b2^\ell, b(k+1)^\ell),$$

where $a, b \in \{1, 2, \ldots, 2^\ell - 1\}$ *and are power-*ℓ *free.*

Proof. Let k, ℓ be integers with $k \geq 2$ and $\ell \geq 1$. By Corollary 1, the only solutions to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = y^{1/\ell}$ in $\mathbb{N}$ are $(x_1, \ldots, x_k, y) = (c\alpha_1^\ell, \ldots, c\alpha_k^\ell, c\beta^\ell)$ where $\alpha_1, \ldots, \alpha_k, \beta, c \in \mathbb{N}$, $\alpha_1 + \cdots + \alpha_k = \beta$, and c is power-ℓ free. Restricted to the set $\{1, 2, \ldots, (k+2)^\ell - 1\}$, we must have $c\alpha_1^\ell, \ldots, c\alpha_k^\ell, c\beta^\ell \leq (k+2)^\ell - 1$. It follows that $\alpha_1^\ell, \ldots, \alpha_k^\ell \in \{1^\ell, 2^\ell, \ldots, (k+1)^\ell\}$ and hence $\alpha_1, \ldots, \alpha_k, \beta \leq k+1$. So $\alpha_1, \ldots, \alpha_k, \beta$ form a solution to $x_1 + \cdots + x_k = y$ in $\{1, 2, \ldots, k+1\}$. Since the only solutions to $x_1 + \cdots + x_k = y$ in $\{1, 2, \ldots, k+1\}$ are

$$(x_1, \ldots, x_{k-1}, x_k, y) = (1, \ldots, 1, 1, k),$$

and

$$(x_1, \ldots, x_{k-1}, x_k, y) = (1, \ldots, 1, 2, k+1),$$

we have either

$$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \beta) = (1, \ldots, 1, 1, 1, k)$$

or

$$(\alpha_1, \ldots, \alpha_{k-1}, \alpha_k, \beta) = (1, \ldots, 1, 2, k+1).$$

Now since $c\beta^\ell \leq (k+2)^\ell - 1$, we have

$$c \leq \frac{(k+2)^\ell - 1}{\beta^\ell} \leq \frac{(k+2)^\ell - 1}{k^\ell} < \left(1 + \frac{2}{k}\right)^\ell \leq 2^\ell.$$

Hence $c \in \{1, 2, \ldots, 2^\ell - 1\}$. □

Proof of Theorem 1. Let $k \geq 2$ and $\ell \geq 1$ be integers. By Lemmas 1 and 2, we have $f(k, \ell) \leq [f(k, 1)]^\ell = (k+2)^\ell$. It remains to show that $f(k, \ell) \geq (k+2)^\ell$. This is true for $\ell = 1$ by Lemma 2. So we assume $\ell \geq 2$. It suffices to show that Breaker wins the $G\big([(k+2)^\ell - 1], k, \ell\big)$ game.

To do this, we build a winning strategy for Breaker based on Lemma 3. If $k = 2$, then Breaker wins the game by the pairing strategy over the sets $\{a, a2^\ell\}$ where $a \in \{1, 2, \ldots, 2^\ell - 1\}$. If $k \geq 3$, then Breaker wins the game by the pairing strategy over the sets $\{a, ak^\ell\}$ and $\{b2^\ell, b(k+1)^\ell\}$ where $a, b \in \{1, 2, \ldots, 2^\ell - 1\}$. In these pairing strategies, if Maker chooses some a or $b2^\ell$ so that $ak^\ell > (k+2)^\ell - 1$ or $b(k+1)^\ell > (k+2)^\ell - 1$, then Breaker arbitrarily chooses an available number in $\{1, 2, \ldots, (k+2)^\ell - 1\}$. □

4. Proof of Theorem 2

We first use the following two lemmas to prove Theorem 2 for $\ell = 1$.

Lemma 4. *For all integers $k \geq 2$, we have $f^*(k, 1) \leq k^2 + 3$.*

Proof. It suffices to show that Maker wins the $G^*([k^2 + 3], k, 1)$ game. For $i = 1, 2, ..., \lceil n/2 \rceil$, let m_i denote the number selected by Maker in round i. For $j = 1, 2, \ldots, \lfloor n/2 \rfloor$, let b_j denote the number selected by Breaker in round j.

We first consider the case that $k = 2$. Then $k^2 + 3 = 7$. Maker starts by choosing $m_1 = 1$. Then no matter what b_1 is, there are three consecutive numbers in $\{2, 3, 4, 5, 6, 7\}$ available to Maker, say $\{a, b, c\}$. Maker sets $m_2 = b$. Notice that $1 + a = b$ and $1 + b = c$. Since Breaker can only choose one of a and c, Maker wins in round 3 by setting $m_3 = a$ or $m_3 = c$.

Now suppose $k = 3$. Then $k^2 + 3 = 12$. Maker starts by choosing $m_1 = 1$. We have 4 cases based on Breaker's choices.

`Case 1:` If $b_1 \neq 2$, then Maker chooses $m_2 = 2$. Suppose Breaker has selected b_2. Now consider the 3-term arithmetic progressions of difference $m_1 + m_2 = 3$:

$$\{3, 6, 9\}, \{4, 7, 10\}, \text{ and } \{5, 8, 11\}.$$

At the start of round 3, Breaker has chosen two numbers and hence one of these 3-term arithmetic progressions is available to Maker. Maker can set m_3 equal to the middle number of the available 3-term arithmetic progression and win the game in round 4 by choosing either the smallest or the largest number of the same 3-term arithmetic progression.

`Case 2:` If $b_1 = 2$, then Maker chooses $m_2 = 3$. Suppose $b_2 \neq 4, 8, 12$. Since $\{4, 8, 12\}$ is a 3-term arithmetic progression of difference $m_1 + m_2 = 4$, Maker can set $m_3 = 8$ and win the game in round 4 by choosing either 4 or 12.

`Case 3:` If $b_1 = 2$, then Maker chooses $m_2 = 3$. Suppose $b_2 = 4$ or 8. Then Maker sets $m_3 = 5$. If $b_3 \neq 9$, then Maker sets $m_4 = 9$. Since $m_1 + m_2 + m_3 = 1 + 3 + 5 = 9 = m_4$, Maker wins the game. Suppose $b_3 = 9$. Then Maker sets $m_4 = 6$. Since $m_1 + m_2 + m_4 = 1 + 3 + 6 = 10$ and $m_1 + m_3 + m_4 = 1 + 5 + 6 = 12$, Maker wins in round 5 by choosing either 10 or 12.

`Case 4:` If $b_1 = 2$, then Maker chooses $m_2 = 3$. Suppose $b_2 = 12$. Then Maker sets $m_3 = 4$. If $b_3 \neq 8$, then Maker sets $m_4 = 8$. Since $m_1 + m_2 + m_3 = 1 + 3 + 4 = 8 = m_4$, Maker wins the game. Suppose $b_3 = 8$. Then Maker sets $m_4 = 5$. Since $m_1 + m_2 + m_4 = 1 + 3 + 5 = 9$ and $m_1 + m_3 + m_4 = 1 + 4 + 5 = 10$, Maker wins in round 5 by choosing either 9 or 10.

Finally, we consider that $k \geq 4$. First notice that, since $k \geq 4$, all the k-sums are at least

$$\sum_{i=1}^{k} i = \frac{1}{2}k^2 + \frac{1}{2}k > 2k.$$

To see this, consider the following strategy for Maker: if a k-sum is available to Maker, then Maker chooses the k-sum and wins the game; otherwise Maker selects the smallest number available. By this strategy, Maker will choose the smallest numbers possible for the first k rounds and the smallest k-sum is $m_1 + \cdots + m_k$.

Also notice that $m_i \leq 2i - 1$ for $i = 1, ..., k$. Indeed, at the start of round i, Maker and Breaker have together chosen $2(i-1) = 2i - 2$ numbers. Hence, one of the numbers in $\{1, 2, \ldots, 2i-1\}$ is still available to Maker. So by Maker's strategy, we have $m_i \leq 2i - 1$.

Since $m_i \leq 2i - 1$ for $i = 1, ..., k$, we have

$$\sum_{i=1}^{k} m_i \leq 1 + 3 + \cdots + 2k - 1 = k^2 \leq k^2 + 3.$$

If Breaker did not choose $m_1 + \cdots + m_k$ during the first k rounds, then Maker chooses $m_1 + \cdots + m_k$ in round $k + 1$ and wins the game.

Now suppose that Breaker has selected $m_1 + \cdots + m_k$ during the first k rounds. Consider the middle of round $k + 1$ when Maker has chosen $k + 1$ numbers but Breaker has only chosen k numbers where s, $1 \leq s \leq k$, of them are k-sums. Since there are $2k+1$ numbers in $\{1, 2, \ldots, 2k+1\}$ and Breaker has chosen only k numbers, we have $m_{k+1} \leq 2k + 1$ by Maker's strategy. Since $m_1, \ldots, m_{k+1}$ are distinct, the total number of k-sums is $\binom{k+1}{k} = k + 1$.

Notice that if Breaker has chosen s k-sums during the first k rounds and one of them is $\sum_{i=1}^{k} m_i$, then

$$m_{k+1-s+j} \leq 2(k + 1 - s + j) - 1 - j = 2(k + 1 - s) + j - 1$$

for $j = 1, 2, \ldots, s$. Indeed, since the k-sums are greater than $2k$, if Breaker has chosen s k-sums, then Breaker has chosen at most $k - s$ numbers in $\{1, 2, \ldots, 2k - s+1\}$. By Maker's strategy, Maker has chosen $k+1$ numbers in $\{1, 2, \ldots, 2k-s+1\}$. If $s = 1$, then we have $m_{k+1} \leq 2k$. If $s > 1$, then by Maker's strategy, we have $m_{k+1} > m_k > \cdots > m_{k+1-s+1}$. Since $m_{k+1}, \ldots, m_{k+1-s+1} \in \{1, 2, \ldots, 2k - s + 1\}$, this is also true.

Now we split it into two cases based on the value of s and what Breaker chooses in round $k + 1$.

`Case 1:` $1 \leq s \leq k - 1$ or $s = k$ and Breaker does not choose a k-sum in round $k + 1$. Then Breaker will have chosen at most k k-sums at the beginning of round $k + 2$. Since $m_i \leq 2i - 1$ for $i = 1, ..., k$ and $m_{k+1-s+j} \leq 2(k + 1 - s) + j - 1$ for $j = 1, 2, \ldots, s$, at the beginning of round $k + 2$, there exists an unclaimed k-sum whose value is at most

$$\begin{aligned}\sum_{i=1}^{k+1-s-2} m_i + \sum_{i=k+1-s}^{k+1} m_i \leq & \sum_{i=1}^{k+1-s-2}(2i-1) + \sum_{j=0}^{s}[2(k+1-s)+j-1] \\ =&(k-s-1)^2 + (s+1)2(k+1-s) + \frac{s(s-1)}{2} - 1 \\ =&k^2 - \frac{1}{2}s^2 + \frac{3}{2}s + 2 \leq k^2 + 3.\end{aligned}$$

Hence Maker chooses this k-sum in round $k+2$ and wins the $G^*([k^2+3], k, 1)$ game.

`Case 2:` $s = k$ and Breaker chooses a k-sum in round $k+1$. In this cases, at the end of round $k+1$, Breaker has chosen all possible k-sums from $\{m_1, \ldots, m_{k+1}\}$. Recall that the k-sums are greater than $2k$. Since $k+2 \leq 2k$ for $k \geq 2$, Breaker did not choose any number in $\{1, 2, \ldots, k+2\}$. So $m_i = i$ for $i = 1, 2, \ldots, k+2$. Notice that the largest k-sum before round $k+2$ is

$$\sum_{i=2}^{k+1} m_i = \sum_{i=1}^{k+1} i = \frac{(k+1)(k+2)}{2} - 1 = \frac{1}{2}k^2 + \frac{3}{2}k.$$

Setting $m_{k+2} = k+2$, Maker now has two larger k-sums which are untouched by Breaker:

$$m_{k+2} + \sum_{i=2}^{k} m_i = k+2 + \frac{k(k+1)}{2} - 1 = \frac{1}{2}k^2 + \frac{3}{2}k + 1$$

and

$$m_{k+1} + m_{k+2} + \sum_{i=2}^{k-1} m_i = k+1+k+2 + \frac{(k-1)k}{2} - 1 = \frac{1}{2}k^2 + \frac{3}{2}k + 2.$$

Since $k \geq 4$, we have

$$k^2 + 3 \geq \frac{1}{2}k^2 + \frac{3}{2}k + 2.$$

Hence Maker can win the $G^*([k^2+3], k, 1)$ game in round $k+3$. $\square$

Lemma 5. *For all integers $k \geq 2$, we have $f^*(k, 1) \geq k^2 + 3$.*

Proof. It suffices to show that Breaker wins the $G([k^2+2], k, 1)$ game. For $i = 1, 2, \ldots, \lceil n/2 \rceil$, let m_i denote the number selected by Maker in round i. For $j = 1, 2, \ldots, \lfloor n/2 \rfloor$, let b_j denote the number selected by Breaker in round j.

We first consider $k = 2$. Then $k^2 + 2 = 2^2 + 2 = 6$. If $m_1 = 1$, then Breaker chooses $b_1 = 4$. Now Breaker wins by the pairing strategy over $\{2, 3\}$ and $\{5, 6\}$. If $m_1 \neq 1$, then Breaker chooses $b_1 = 1$. Now there are only two solutions available to Maker: $2+3=5$ and $2+4=6$. There are three cases.

`Case 1:` $m_1 = 2$. Then Breaker wins by the pairing strategy over $\{3, 5\}$ and $\{4, 6\}$.

`Case 2:` $m_1 \neq 1, 2$, $b_1 = 1$, $m_2 = 2$. Then Breaker wins by the pairing strategy over $\{3, 5\}$ and $\{4, 6\}$.

`Case 3:` $m_1 \neq 1, 2$, $b_1 = 1$, $m_2 \neq 2$. Then by choosing $b_2 = 2$, Breaker wins because the smallest numbers now available to Maker are 3 and 4, and $3+4 = 7 > 6$.

Now we consider $k \geq 3$. Notice that we have $k^2 - 1 \geq 2k + 2$ when $k \geq 3$. We will prove that Breaker wins with the following strategy:

(1) in each round $i \in [k-1]$, Breaker chooses smallest number available;

(2) and in round k, if there is an unclaimed number in $[2k-2]$, then Breaker chooses the unclaimed number; otherwise, Breaker's strategy depends on the sum of the numbers in $[2k-2]$ claimed by Maker, which is denoted by S:

- If $S = (k-1)^2 + 3$, then Breaker chooses the smallest numbers possible.
- If $S = (k-1)^2 + 2$, then Breaker plays the pairing strategy over $\{2k - 1, k^2 + 2\}$.
- If $S = (k-1)^2 + 1$, then Breaker plays the pairing strategy over $\{2k - 1, k^2 + 1\}$ and $\{2k, k^2 + 2\}$.
- If $S = (k-1)^2$, then Breaker plays the pairing strategy over $\{2k-1, k^2\}$, $\{2k, k^2 + 1\}$, and $\{2k + 1, k^2 + 2\}$.

Let $a_1 < a_2 < a_3 < \cdots < a_s$ with $s \leq \lceil n/2 \rceil$ be the numbers chosen by Maker when the game ends. We claim the following hold:

(i) $a_i \geq 2i - 1$ for $i = 1, 2, \ldots, k$, $a_{k+1} \geq 2k$, and $a_{k+2} \geq 2k + 1$;

(ii) if $a_{k-1} > 2k - 2$, then Breaker wins;

(iii) the smallest k-sum possible for Maker is $\sum_{i=1}^{k} a_i \geq \sum_{i=1}^{k}(2i-1) = k^2$ and hence Maker needs one of k^2, $k^2 + 1$, and $k^2 + 2$ to win;

(iv) if a k-sum does not contain all $\{a_1, ..., a_{k-1}\}$, then Breaker wins.

Here is why (i) holds. Since $a_i \geq 1 = 2 \cdot 1 - 1$, this is true for $i = 1$. Now consider $2 \leq i \leq k$. By Breaker's strategy, Breaker can select at least $i - 1$ numbers in $\{1, 2, \ldots, 2(i-1)\}$. So Maker can select at most $i-1$ numbers in $\{1, 2, \ldots, 2(i-1)\}$. Hence $a_i \geq 2(i-1) + 1 = 2i - 1$.

To see that (ii) holds, notice that if $a_{k-1} > 2k - 2$, then $a_{k-1} \geq 2k - 1$ and $a_k \geq 2k$. Hence the smallest k-sum possible for Maker is

$$\sum_{i=1}^{k} a_i \geq 2k - 1 + 2k + \sum_{i=1}^{k-2}(2i-1) = 2k - 1 + 2k + (k-2)^2 = k^2 + 3 > k^2 + 2$$

and hence Breaker wins.

The reason (iv) holds is because if a k-sum does not contain all of $\{a_1, \ldots, a_{k-1}\}$, then the k-sum is at least

$$a_k + a_{k+1} + \sum_{i=1}^{k-2} a_i \geq 2k - 1 + 2k + (k-2)^2 = k^2 + 3 > k^2 + 2.$$

We first suppose that after Maker has chosen $m_1, \ldots, m_k$, there is an unclaimed number in $[2k-2]$. In this case, Breaker sets b_k equal to some number in $[2k-2]$. Now Breaker has chosen k numbers in $[2k-2]$ which implies that Maker can choose at most $k-2$ numbers in $[2k-2]$. Hence $a_{k-1} > 2k-2$. It follows that, Breaker wins.

Now assume that all the numbers in $[2k-2]$ are claimed in the middle of round k when Breaker has chosen k numbers and Breaker has chosen $k-1$ numbers. In this case, we must have $a_1, \ldots, a_{k-1} \in [2k-2]$ and hence $\sum_{i=1}^{k-1} a_i = S$. We consider the solutions to $x_1 + \cdots + x_k = y$, where $x_1, \ldots, x_k$ are distinct, such that Breaker has not occupied any number in them. Recall that if a k-sum does not contain all numbers in $\{a_1, \ldots, a_{k-1}\}$, then Breaker wins. So we have the following cases.

`Case 1:` If $S = \sum_{i=1}^{k-1} a_i = (k-1)^2$, then there are three solutions to $x_1 + \cdots + x_k = y$, where $x_1, \ldots, x_k$ are distinct, such that Breaker has not occupied any number in them: $\{a_1, \ldots, a_{k-1}, 2k-1, k^2\}$, $\{a_1, \ldots, a_{k-1}, 2k, k^2+1\}$, and $\{a_1, \ldots, a_{k-1}, 2k+1, k^2+2\}$. This is because if $S = \sum_{i=1}^{k-1} a_i = (k-1)^2$, then

$$a_k + \sum_{i=1}^{k-1} a_i \geq 2k - 1 + (k-1)^2 = k^2,$$

$$a_{k+1} + \sum_{i=1}^{k-1} a_i \geq 2k + (k-1)^2 = k^2 + 1,$$

$$a_{k+2} + \sum_{i=1}^{k-1} a_i \geq 2k + 1 + (k-1)^2 = k^2 + 2,$$

and

$$a_s + \sum_{i=1}^{k-1} a_i \geq 2k + 1 + 1 + (k-1)^2 = k^2 + 3 > k^2 + 2$$

for $s \geq k+3$.

`Case 2:` If $S = \sum_{i=1}^{k-1} a_i = (k-1)^2 + 1$, then there are two solutions to $x_1 + \cdots + x_k = y$, where $x_1, \ldots, x_k$ are distinct, such that Breaker has not occupied any number in them: $\{a_1, \ldots, a_{k-1}, k^2+1\}$ and $\{a_1, \ldots, a_{k-1}, a_{k+1}, k^2+2\}$. This is because if $S = \sum_{i=1}^{k-1} a_i = (k-1)^2 + 1$, then

$$a_k + \sum_{i=1}^{k-1} a_i \geq 2k - 1 + (k-1)^2 + 1 = k^2 + 1,$$

$$a_{k+1}+\sum_{i=1}^{k-1}a_i \geq 2k+(k-1)^2+1=k^2+2,$$

and

$$a_s+\sum_{i=1}^{k-1}a_i \geq 2k+1+(k-1)^2+1=k^2+3>k^2+2$$

for $s \geq k+2$.

`Case 3:` If $S=\sum_{i=1}^{k-1}a_i=(k-1)^2+2$, then there is only one solution to $x_1+\cdots+x_k=y$, where $x_1,\ldots,x_k$ are distinct, such that Breaker has not occupied any number in them: $\{a_1,\ldots,a_k,k^2+2\}$. This is because if $S=\sum_{i=1}^{k-1}a_i=(k-1)^2+2$, then

$$a_k+\sum_{i=1}^{k-1}a_i \geq 2k-1+(k-1)^2+2=k^2+2,$$

and

$$a_s+\sum_{i=1}^{k-1}a_i \geq 2k+(k-1)^2+2=k^2+3>k^2+2$$

for $s \geq k+1$.

In Case 1, Breaker uses the pairing strategy over $\{2k-1,k^2\}$, $\{2k,k^2+1\}$, and $\{2k+1,k^2+2\}$. Since these sets are pairwise disjoint, Breaker wins. Similarly, in Case 2, Breaker uses the pairing strategy over $\{2k-1,k^2+1\}$ and $\{2k,k^2+2\}$; and in Case 3, Breaker uses the pairing strategy over $\{2k-1,k^2+2\}$. □

Proof of Theorem 2. Let k,ℓ be integers with $k \geq 2$ and $\ell \geq 1$. By Lemmas 1, 4 and 5, we have $f^*(k,\ell) \leq [f^*(k,1)]^\ell=(k^2+3)^\ell$. It remains to show that $f^*(k,\ell) \geq (k^2+3)^\ell$ for all $\ell \geq 2$. To do this, it suffices to show that Breaker wins the $G([(k^2+3)^\ell-1],k,\ell)$ game. For all $c \in \{1,2,\ldots,2^\ell-1\}$, let

$$A(c)=\{c\cdot 1^\ell, c\cdot 2^\ell,\ldots,c\cdot(k^2+2)^\ell\}\cap\{1,2,\ldots,(k^2+3)^\ell-1\}.$$

Notice that if $c,c' \in \{1,2,\ldots,2^\ell-1\}$ with $c \neq c'$, then $A(c)\cap A(c')=\emptyset$. By Corollary 1, every solution to $x_1^{1/\ell}+\cdots+x_k^{1/\ell}=y^{1/\ell}$, with $x_1,\ldots,x_k$ distinct, in $\{1,2,\ldots,(k^2+3)^\ell-1\}$ belongs to $A(c)$ for some $c \in \{1,2,\ldots,2^{\ell-1}\}$.

Let $\mathcal{B}$ be a Breaker's winning strategy for the $G^*([k^2+2],k,1)$ game. We define a Breaker's strategy for the $G([(k^2+3)^\ell-1],k,\ell)$ game recursively. For rounds $i=1,2,\ldots$, let m_i be the number chosen by Maker and let b_i be the number chosen by Breaker. Let $m_1=c_1a_1^\ell$ where c_1 is power-ℓ free. If $\mathcal{B}$ tells Breaker to choose α_1 for the $G^*([k^2+2],k,1)$ game given that Maker has selected a_1, then Breaker sets $b_1=c_1\alpha_1^\ell$. Consider round $i \geq 2$. Suppose Maker has chosen $m_1=c_1a_1^\ell, m_2=c_2a_2^\ell,\ldots,m_i=c_ia_i^\ell$ and Breaker has selected $b_1=c_1\alpha_1^\ell, b_2=c_2\alpha_2^\ell,\ldots,b_{i-1}=c_{i-1}\alpha_{i-1}^\ell$. Let $c_{j_1},c_{j_2},\ldots,c_{j_s} \in \{1,\ldots,i-1\}$ be all the indices such that

$$c_{j_1}=c_{j_2}=\cdots=c_{j_s}=c_i.$$

If $\mathcal{B}$ tells Breaker to choose α_i for the $G^*([k^2+2], k, 1)$ game given that Maker has has selected $a_{j_1}, a_{j_2}, \ldots, a_{j_s}$, a_i and Breaker has selected $b_{j_1}, b_{j_2}, \ldots, b_{j_s}$, then Breaker sets $b_i = c_i\alpha_i^\ell$.

Since $\mathcal{B}$ is a winning strategy for Breaker, Breaker can stop Maker from completing a solution set from each $A(c)$ and hence wins the game. □

5. Proof of Theorem 3

The following observation will be use in proving Theorem 3.

Lemma 6. *Let k, ℓ be integers with $k \geq 2$ and $\ell \leq -1$. If $n < 2k^{-\ell}$ and Maker does not choose 1 in the first round, then Breakers wins the $G([n], k, \ell)$ game.*

Proof. Suppose $n < 2k^{-\ell}$ and Maker does not choose 1 in the first round. We show that Breaker wins the $G([n], k, \ell)$ game by choosing 1 in the first round. Suppose, for a contradiction, that Maker wins. Let $(x_1, \ldots, x_k, y) = (a_1, \ldots, a_k, b)$ be a solution to Equation (1) in $\{1, 2, \ldots, n\}$ completed by Maker. Then since $a_i \leq n < 2k^{-\ell}$ for all $i = 1, \ldots, k$, we have

$$b^{1/\ell} = a_1^{1/\ell} + \cdots + a_k^{1/\ell} > k(2k^{-\ell})^{1/\ell} = 2^{1/\ell}.$$

So $b < 2$ which is impossible. □

Proof of Theorem 3. We first prove that, if $k \geq 1/(2^{-1/\ell} - 1)$, then $f(k, \ell) \geq (k+1)^{-\ell}$. To do this, it suffices to show that that Breaker wins the $G([(k+1)^{-\ell}-1], k, \ell)$ game. By straightforward calculation, we have

$$(k+1)^{-\ell} - 1 < 2k^{-\ell}.$$

Hence, by Lemma 6, we can assume that Maker chooses 1 in the first round and $b = 1$. Now we show that the only solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = 1$ in $\{1, 2, \ldots, (k+1)^{-\ell}-1\}$ is $(x_1, \ldots, x_k) = (k^{-\ell}, \ldots, k^{-\ell})$. This would imply that Breaker can choose $k^{-\ell}$ in the first round and win the game. Let $a_1, \ldots, a_k \in \{1, 2, \ldots, (k+1)^{-\ell} - 1\}$ with

$$a_1^{1/\ell} + \cdots + a_k^{1/\ell} = 1,$$

and $a_1 \leq \cdots \leq a_k$. Since the sum a rational number and an irrational number is irrational, $a_1^{1/\ell}, \ldots, a_k^{1/\ell}$ are rational numbers. Since $a_1, \ldots, a_k \in \{1, 2, \ldots, (k+1)^{-\ell} - 1\}$, we have $a_1, \ldots, a_k \in \{1, 2^{-\ell}, \ldots, k^{-\ell}\}$. If $a_i < k^{-\ell}$ for some $i \in [k]$, then

$$1 = a_1^{1/\ell} + \cdots + a_k^{1/\ell} > k(k^{-\ell})^{1/\ell} = 1$$

which is impossible. Hence the only solution to $x_1^{1/\ell} + \cdots + x_k^{1/\ell} = 1$ in $\{1, 2, \ldots, (k+1)^{-\ell}-1\}$ is $(x_1, \ldots, x_k) = (k^{-\ell}, \ldots, k^{-\ell})$ and Breaker wins the $G([(k+1)^{-\ell}-1], k, \ell)$ game.

Now we prove that if $k \geq 4$, then $f(k,\ell) \leq (k+2)^{-\ell}$. By Lemma 1, $f(k,\ell) \leq [f(k,-1)]^{-\ell}$. Hence, it suffices to show that for all $k \geq 4$, $f(k,-1) \leq k+2$. We split it into two cases.

`Case 1:` $k+1 \neq p$ or p^2 for any prime p. We will prove that $f(k,-1) \leq k+1$. To do this, we will prove that Maker wins the $G([k+1],k,-1)$ game. In this case, we have $k+1 = rs$ for some integers $r > 1$ and $s > 1$ with $r \neq s$. Then we have $(r-1)s \neq r(s-1)$, $(r-1)s < k < k+1$ and $r(s-1) < k < k+1$. Consider the following solutions in $\{1,2,\ldots,k+1\}$:

$$(x_1,x_2,\ldots,x_{k-1},x_k,y) = (k,k,\ldots,k,k,1),$$

$$(x_1,\ldots,x_{(r-1)s},x_{(r-1)s+1},\ldots,x_k,y) = (rs,\ldots,rs,r(s-1),\ldots,r(s-1),1),$$

and

$$(x_1,\ldots,x_{r(s-1)},x_{r(s-1)+1},\ldots,x_k,y) = (rs,\ldots,rs,(r-1)s,\ldots,(r-1)s,1).$$

Based on these solutions, Maker wins the $G([k+1],k,-1)$ game using the following strategy: Maker chooses 1 in the first round; if Breaker does not choose k in the first round, then Maker chooses k in the second round to win the game; otherwise, Maker will choose $k+1 = rs$ in the second round and win the game by choosing either $r(s-1)$ or $(r-1)s$ in the third round.

`Case 2:` $k+1 = p$ or p^2 for some prime $p \geq 5$. We will show that Maker wins the $G([k+2],k,-1)$ game.

Since $k+1 \geq 5$ is odd, k is even and $k \geq 4$. Hence $(k+2)/2 \neq k$. Consider the following solutions in $\{1,2,\ldots,k+2\}$:

$$(x_1,x_2,\ldots,x_{k-1},x_k,y) = (k,k,\ldots,k,k,1),$$

$$(x_1,\ldots,x_{(k-2)/2},x_{(k-2)/2+1},\ldots,x_k,y) = (k-2,\ldots,k-2,k+2,\ldots,k+2,1),$$

and

$$(x_1,x_2,x_3,\ldots,x_k,y) = ((k+2)/2,(k+2)/2,k+2,\ldots,k+2,1).$$

Based on these solutions, Maker wins the $G([k+2],k,-1)$ game by the following strategy: Maker chooses 1 in the first round; if Breaker does not choose k in the first round, then Maker chooses k in the second round to win the game; otherwise, Maker will choose $k+2$ in the second round and win the game by choosing either $(k+2)/2$ or $k-2$ in the third round. □

5.1. Remarks

In the proof of Theorem 3, we showed that if $k+1 = p$ or p^2 for some prime $p \geq 5$, then $f(k,-1) \leq k+2$. This inequality becomes equality when $k+1 = p$ for some odd prime p.

Theorem 7. *If $k+1=p$ for some odd prime p, then $f(k,-1)=k+2$.*

Proof. Suppose $k+1=p$ for some odd prime. By Theorem 3, we have $f(k,-1) \leq k+2$. It remains to show that $f(k,-1) \geq k+2$. To do this, it suffices to show that Breaker wins the $G([k+1],k,-1)$ game. We consider two cases.

`Case 1:` $k+1=3$. The only solution to $1/x_1+\cdots+1/x_k=1/y$ in $\{1,2,3\}$ with $x_1,\ldots,x_k$ not necessarily distinct is $(x_1,x_2,y)=(2,2,1)$. Hence Breaker can win by choosing either 1 or 2 in the first round.

`Case 2:` $k+1 \geq 5$. By Lemma 6, if Maker does not choose 1 in the first round, then Breaker wins. So we assume that Maker chooses 1 in the first round. Now we show that Breaker wins by choosing k in the first round. It suffices to show that $\{1,2,\ldots,k-1,k+1\}$ does not have a solution to $1/x_1+\cdots+1/x_k=1/1$ where x_1, …, x_k are not necessarily distinct. Suppose $(x_1,x_2,\ldots,x_{k-1},x_k)=(a_1,a_2,\ldots,a_{k-1},a_k)$ is a solution in $\{1,2,\ldots,k-1,k+1\}$. We show that $a_k=k+1$. Suppose not. Then $a_i<k$ for all $i=1,2,\ldots,k$. So

$$\frac{1}{a_1}+\cdots+\frac{1}{a_k} > \frac{1}{k}+\cdots+\frac{1}{k}=\frac{1}{1}$$

which is a contradiction. Hence $a_k=k+1$. Now we have

$$1=\frac{r}{k+1}+\sum_{i=1}^{k-r}\frac{1}{a_i}$$

where $r \in \{1,2,\ldots,k-1\}$ and $a_i<k$ for all $i=1,\ldots,k-r$. Rearranging the equation, we get

$$\sum_{i=1}^{k-r}\frac{1}{a_i}=\frac{p-r}{p}.$$

Since p is prime, p divides the least common multiple of $a_1,\ldots,a_{k-r}$. Since p is prime, p divides a_i for some i which is a contradiction because $a_i<p$ for all i. Hence Breaker wins the game. □

We are unable to verify that $f(k,-1)=k+2$ when $k+1=p^2$ for some odd prime p. However, we believe this should be the case.

Conjecture 2. If $k+1=p^2$ for some odd prime p, then $f(k,-1)=k+2$.

6. Proof of Theorem 4

To prove Theorem 4, we need the following result.

Lemma 7. *Let $k \geq 4$ be an integer and let $A = \{1, \ldots, 2k+1\} \cup \{k^2-k+1, \ldots, k^2+2k\}$. Then Maker wins the $G^*(A, x_1 + \cdots + x_k = y)$ game.*

Proof. Let $k \geq 4$. For $i = 1, \ldots, k+3$, let m_i be the number selected by Maker in round i and let b_i be the number selected by Breaker in round i.

Consider the following strategy for Maker:

(1) Set $m_1 = 1$ and $M_1 = \{\{2,3\}, \{4,5\}, \ldots, \{2k, 2k+1\}\}$.

(2) For $i = 2, \ldots, k+1$, if $b_{i-1} \in B$ for some $B \in M_{i-1}$, then set $m_i \in B \backslash \{b_{i-1}\}$ and $M_i = M_{i-1} \backslash \{B\}$; if $b_{i-1} \notin B$ for any $B \in M_{i-1}$, then set $m_i = \min_{S \in M_{i-1}} \min S$, $M_i = M_{i-1} \backslash S'$ where $m_i \in S'$.

(3) In round $k+2$, if there exists a subset $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+1}\}$ of size k such that $a_1 + \cdots + a_k \in \{k^2-k+1, \ldots, k^2+2k\} \backslash \{b_1, \ldots, b_{k+1}\}$, then set $m_{k+2} = a_1 + \cdots + a_k$. Otherwise, set $m_{k+2} = 2k+1$, and then, in round $k+3$, set $m_{k+3} = a_1 + \cdots + a_k$ where $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+2}\}$ has size k with $a_1 + \cdots + a_k \in \{k^2-k+1, \ldots, k^2+2k\} \backslash \{b_1, \ldots, b_{k+2}\}$.

In Step (3), Maker wins for the first case. So it remains to show that if no subset $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+1}\}$ of size k satisfies $a_1 + \cdots + a_k \in \{k^2-k+1, \ldots, k^2+2k\} \backslash \{b_1, \ldots, b_{k+1}\}$, then Maker can set $m_{k+2} = 2k+1$ in round $k+2$ and there exists a subset $\{a_1, ..., a_k\} \subseteq \{m_1, \ldots, m_{k+2}\}$ of size k such that $a_1 + \cdots + a_k \in \{k^2-k+1, \ldots, k^2+2k\} \backslash \{b_1, \ldots, b_{k+2}\}$.

Suppose, at the beginning of round $k+2$, no subset $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+1}\}$ of size k satisfies $a_1 + \cdots + a_k \in \{k^2-k+1, \ldots, k^2+2k\} \backslash \{b_1, \ldots, b_{k+1}\}$. First note that by Maker's strategy, for all $i = 2, \ldots, k+1$, $m_i = 2(i-1)$ or $2(i-1)+1$. So for all subsets $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+1}\}$ of size k, we have

$$a_1 + \cdots + a_k \geq 1 + 2 + 4 + \cdots + 2(k-1) = k^2 - k + 1$$

and

$$a_1 + \cdots + a_k \leq 3 + 5 + \cdots + 2k + 1 = (k+1)^2 - 1 = k^2 + 2k.$$

So if no subset $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+1}\}$ of size k satisfies $a_1 + \cdots + a_k \in \{k^2-k+1, \ldots, k^2+2k\} \backslash \{b_1, \ldots, b_{k+1}\}$, then $b_1, \ldots, b_{k+1} \notin \{1, \ldots, 2k+1\}$. Now according to Maker's strategy, we have, $m_1 = 1$, and $m_i = 2(i-1)$ for all $i = 2, \ldots, k+1$. This implies that at the beginning of round $k+2$, $2k+1$ is available to Maker and hence Maker can set $m_{k+2} = 2k+1$. At the same time, for all subsets $\{a_1, \ldots, a_k\} \subseteq \{m_1, \ldots, m_{k+1}\}$ of size k, we have $a_1 + \cdots + a_k \leq 2 + 4 + \cdots + 2k = k^2 + k$ and hence $b_1, \ldots, b_{k+1} \leq k^2 + k$. By setting $m_{k+2} = 2k+1$, there are at least two subsets of $\{m_1, \ldots, m_{k+2}\}$ of size k whose sum is greater than $k^2 + k$. They are $\{2, 4, \ldots, 2(k-1), 2k+1\}$ and $\{2, 4, \ldots, 2(k-2), 2k, 2k+1\}$. The first subset sums to $k^2 + k + 1 < k^2 + 2k$ and the second one sums to $k^2 + k + 3 < k^2 + 2k$.

Since Breaker can only occupy one of them in round $k+2$, there exists a subset $\{a_1, \ldots, a_k\} \subseteq \{m_1, ..., m_{k+2}\}$ of size k such that $a_1 + \cdots + a_k \in \{k^2 - k + 1, \ldots, k^2 + 2k\} \backslash \{b_1, \ldots, b_{k+2}\}$. This proves that Maker wins the $G^*(A, x_1 + \cdots + x_k = y)$ game. $\square$

Proof of Theorem 4. By Lemma 1, we have $f^*(k, \ell) \leq [f^*(k, -1)]^{-\ell}$. It remains to show that $f^*(k, -1) = \exp(O_k(k \log k))$. By Theorem 6, it suffices to find a finite set $A \subseteq \mathbb{N}$ such that Maker wins the $G^*(A, x_1 + \cdots + x_k = y)$ game and the least common multiple of A is small.

Let $k \geq 4$ be an integer and let $A := \{1, \ldots, 2k+1\} \cup \{k^2 - k + 1, \ldots, k^2 + 2k\}$. By Theorem 6 and Lemma 7, we have

$$\begin{aligned} f^*(k, -1) \leq & \operatorname{lcm}\{n : n \in A\} \\ \leq & \operatorname{lcm}\{1, ..., 2k+1\} \operatorname{lcm}\{k^2 - k + 1, ..., k^2 + 2k\} \\ \leq & \operatorname{lcm}\{1, ..., 2k+1\}(k^2 + 2k)^{3k} \\ = & e^{(2+o_k(1))k} e^{3k \log(k^2+2k)}. \end{aligned}$$

Hence we have $f^*(k, -1) = \exp(O_k(k \log k))$. $\square$

6.1. Remarks

By exhaustive search, we are able to find the exact value of $f^*(k, -1)$ for $k = 2$.

Proposition 1. *We have* $f^*(2, -1) = 36$.

Proof. We first show that Maker wins the $G^*([36], 2, -1)$ game. Consider the following solutions to $1/x_1 + 1/x_2 = 1/y$ in $\{1, 2, \ldots, 36\}$ with $x_1 \neq x_2$: $(x_1, x_2, y) = (4, 12, 3)$, $(6, 12, 4)$, $(12, 36, 9)$, and $(18, 36, 12)$.

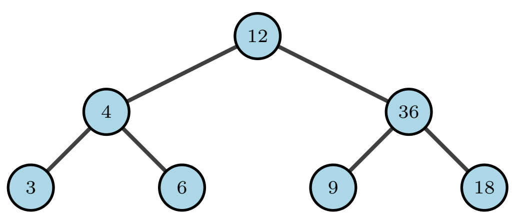

Figure 1: Rooted Binary Tree for Solutions to $1/x_1 + 1/x_2 = 1/y$

In Figure 1, we constructed a rooted binary tree based on these solutions. Each path from the root 12 to a leaf is a solution set to $1/x_1 + 1/x_2 = 1/y$. It is easy to see that Maker can win this game by doing the following:

(1) Maker selects the root in round 1.

(2) In round 2, Maker selects a vertex that is adjacent to the root such that both of its children are untouched by Breaker.

(3) In round 3, Maker chooses a child of the vertex that Maker selected in round 2.

Now we show that Breaker wins the $G^*([35], 2, -1)$ game. By standard calculation, one can check that there are 13 solutions to $1/x_1 + 1/x_2 = 1/y$ in [35]: $\{2, 3, 6\}$, $\{3, 4, 12\}$, $\{4, 6, 12\}$, $\{4, 5, 20\}$, $\{5, 6, 30\}$, $\{6, 8, 24\}$, $\{6, 9, 18\}$, $\{6, 10, 15\}$, $\{8, 12, 24\}$, $\{10, 14, 35\}$, $\{10, 15, 30\}$, $\{12, 20, 30\}$, and $\{12, 21, 28\}$. Breaker wins the game using the pairing strategy over $\{4, 12\}$, $\{8, 24\}$, $\{10, 15\}$, $\{2, 3\}$, $\{5, 20\}$, $\{6, 30\}$, $\{9, 18\}$, $\{14, 35\}$, $\{20, 30\}$, and $\{21, 28\}$. $\square$

For general k, Theorem 4 only provides an upper bound for $f^*(k, -1)$. It is trivially true that $f^*(k, -1) \geq 2k + 1$ because Maker needs to occupy at least $k + 1$ numbers to win. However, we do not have a nontrivial lower bound.

Problem 1. Find a nontrivial lower bound for $f^*(k, -1)$.

7. Equations with Arbitrary Coefficients

In this section, we briefly discuss the Maker-Breaker Rado games for the equation

$$a_1x_1 + \cdots + a_kx_k = y, \tag{4}$$

where $k, a_1, \ldots, a_k$ are positive integers with $k \geq 2$ and $a_1 \geq a_2 \geq \cdots \geq a_k$. Write $w := a_1 + \cdots + a_k$, and $w^* := \sum_{i=1}^{k}(2i - 1)a_i$. Let $f(a_1, \ldots, a_k; y)$ be the smallest positive integer n such that Maker wins the $G([n], a_1x_1 + \cdots + a_kx_k = y)$ game and let $f^*(a_1, \ldots, a_k; y)$ be the smallest positive integer n such that Maker wins the $G^*([n], a_1x_1 + \cdots + a_kx_k = y)$ game.

Hopkins and Schaal [16], and Guo and Sun [11], proved that if $\{1, 2, \ldots, a_kw^2 + w - a_k\}$ is partitioned into two classes, then one of them contains a solution to Equation (4) with $x_1, \ldots, x_k$ not necessarily distinct; and there exists a partition of $\{1, 2, \ldots, a_kw^2 + w - a_k - 1\}$ into two classes such that neither contains a solution to Equation (4) with $x_1, \ldots, x_k$ not necessarily distinct. By these results and strategy stealing, we have $f(a_1, \ldots, a_k; y) \leq a_kw^2 + w - a_k$. The strategy stealing argument here is similar to the one in Section 1 where we explained that $f(k, \ell) \leq R(k, \ell)$ and $f^*(k, \ell) \leq R^*(k, \ell)$. The next theorem shows that, in fact, $f(a_1, \ldots, a_k; y)$ is much smaller than $a_kw^2 + w - a_k$.

Theorem 8. *For all integers $k \geq 2$, we have $w + 2a_k \leq f(a_1, \ldots, a_k; y) \leq w + a_{k-1} + a_k$.*

Proof. The case that $k = 2$ and $a_1 = a_2 = 1$ is a special case of Lemma 2. So we assume that $k > 2$ or $k = 2$ but $a_1 \geq 2$. Then $w > 2$.

We first show that Maker wins the $G([w + a_{k-1} + a_k], a_1x_1 + \cdots + a_kx_k = y)$ game. Maker chooses 1 in round 1. If Breaker does not choose w in round 1, then Maker wins in round 2 by choosing w. If Breaker chooses w in round 1, then Maker chooses 2 in round 2 and either $w+a_k$ or $w+a_{k-1}+a_k$ in round 3 to win the game.

Now we show that Breaker wins the $G([w+2a_k-1], a_1x_1+\cdots+a_kx_k = y)$ game. The only solutions to Equation (4) in $\{1, 2, \dots, w + 2a_k - 1\}$ are

$$(x_1, x_2, \dots, x_{k-1}, x_k, y) = (1, 1, \dots, 1, 1, w)$$

and

$$(x_1, x_2, \dots, x_{k-1}, x_k, y) = (1, 1, \dots, 1, 2, w + a_k).$$

Now Breaker wins by the pairing strategy over $\{1, w\}$ and $\{2, w + a_k\}$. Note that if $a_i = a_k$ for some $i \in \{1, 2, \dots, k - 1\}$, then $(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_k, y) = (1, \dots, 1, 2, 1, \dots, 1, w + a_1)$ is also a solution, but Breaker can still win the game by the pairing strategy becuase $w + a_i = w + a_k$. $\square$

The next theorem provides lower and upper bounds for $f^*(a_1, \dots, a_k; y)$.

Theorem 9. *For all integers $k \geq 4$, we have*

$$w^* \leq f^*(a_1, \dots, a_k; y) \leq w^* + (2k - 2)(a_1 - a_k) + (k + 3)a_{k-2}.$$

Proof. Let $k \geq 4$ be an integer and write $W = w^* + (2k-2)(a_1 - a_k) + (k+3)a_{k-2}$. We first show that Breaker wins the $G^*([w^* - 1], a_1x_1 + \cdots + a_kx_k = y)$ game by choosing the smallest number available each round. Suppose, for a contradiction, that Maker wins. Let $\alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_s$, where $s \geq k+1$, be the numbers chosen by Maker after winning the game. Then by Breaker's strategy, we have $\alpha_i \geq 2i - 1$ for all $i = 1, 2, \dots, k$. By the rearrangement inequality [13], the smallest k-sum is

$$\sum_{i=1}^{k} a_i\alpha_i \geq \sum_{i=1}^{k}(2i - 1)a_i = w^*$$

which is a contradiction.

Now we show that Maker wins the $G^*([W], a_1x_1 + \cdots + a_kx_k = y)$ game. We split it into two cases.

`Case 1:` $\alpha_1 = \alpha_k = c$ for some c. Since the coefficients of $x_1, \dots, x_k$ are the same, Maker's strategy defined in the proof of Lemma 4 still applies by multiplying the k-sums in the proof of Lemma 4 by c. So Maker wins the $G^*([ck^2 + 3c], a_1x_1 + \cdots + a_kx_k = y)$ game. Since

$$W = w^* + (2k - 2)(a_1 - a_k) + (k + 3)a_{k-2} = ck^2 + ck + 3c > ck^2 + 3c,$$

Maker wins the $G^*([W], a_1x_1 + \cdots + a_kx_k = y)$ game.

`Case 2:` $a_1 > a_k$. We will show that Maker wins the game with the following strategy:

(1) Maker chooses the smallest number available each round for the first $k+1$ rounds;

(2) and then chooses an available k-sum in round $k+2$.

For $i = 1, 2, \ldots, k+1$, let m_i be the number chosen by Maker in round i. Then by Maker's strategy, we have $i \le m_i \le 2i-1$ for all $i = 1, 2, \ldots, k+1$.

Since $a_1 > a_k$, there exists $t \in \{2, 3, \ldots, k\}$ such that $\alpha_t < \alpha_{t-1}$. For $i = 1, \ldots, k+1$, let m_i be the number chosen by Maker in round i. By the rearrangement inequality, we have the following k distinct k-sums involving only $m_1, \ldots, m_k$:

$$(a_t m_{t+j} + a_{t+j} m_t) - (a_t m_t + a_{t+j} m_{t+j}) + \sum_{i=1}^{k} a_i m_i, \text{ where } j = 0, 1, \ldots, k-t$$

and

$$(a_{t-j'} m_k + a_k m_{t-j'}) - (a_{t-j'} m_{t-j'} + a_k m_k) + \sum_{i=1}^{k} a_i m_i, \text{ where } j' = 1, 2, \ldots, t-1.$$

Among these distinct k-sums, the smallest is $\sum_{i=1}^{k} a_i m_i$ and the largest is

$$(a_1 m_k + a_k m_1) - (a_1 m_1 + a_k m_k) + \sum_{i=1}^{k} a_i m_i = a_1 m_k + \left(\sum_{i=2}^{k-1} a_i m_i\right) + a_k m_1. \quad (5)$$

Since $k \ge 4$, there are two terms of the form $a_i m_i$, $i \in \{2, \ldots, k-1\}$, in the middle of the right hand side of Equation (5). Replacing m_{k-1} with m_{k+1} and replacing m_{k-2} with m_{k+1}, we get two larger and distinct k-sums:

$$a_1 m_k + \left(\sum_{i=2}^{k-2} a_i m_i\right) + a_{k-1} m_{k+1} + a_k m_1$$

and

$$a_1 m_k + \left(\sum_{i=2}^{k-3} a_i m_i\right) + a_{k-2} m_{k+1} + a_{k-1} m_{k-1} + a_k m_1.$$

The largest of these k-sums is

$$\begin{aligned}
&a_1m_k + \left(\sum_{i=2}^{k-3} a_im_i\right) + a_{k-2}m_{k+1} + a_{k-1}m_{k-1} + a_km_1\\
=&a_1m_k + +a_{k-2}m_{k+1} + a_km_1 - a_1m_1 - a_{k-2}m_{k-2} - a_km_k + \sum_{i=1}^{k} a_im_i\\
=&(m_k - m_1)(a_1 - a_k) + a_{k-2}(m_{k+1} - m_{k-2}) + \sum_{i=1}^{k} a_im_i\\
\le&w^* + [(2k-1)-1](a_1 - a_k) + [2k+1-(k-2)]a_{k-2}\\
=&w^* + (2k-2)(a_1 - a_k) + (k+3)a_{k-2} = W.
\end{aligned}$$

So there exists a k-sum unoccupied by Breaker in the beginning of round $k+2$ and hence Maker wins the $G^*([W], a_1x_1 + \cdots + a_kx_k = y)$ game by choosing the available k-sum in round $k+2$. □

The bounds in Theorem 9 can be optimized using the technique in the proofs of Lemmas 4 and 5, but we do not attempt it here.

8. Concluding Remarks

It would be interesting to study Rado games for other well-studied equations in arithmetic Ramsey theory. One direction is to study Rado games for

$$a_1x_1^{1/\ell} + \cdots + a_kx_k^{1/\ell} = y^{1/\ell}, \tag{6}$$

where $\ell, k, a_1, \ldots, a_k$ are positive integers with $k \ge 2$ and $\ell \ne 0$. We studied the $G([n], a_1x_1 + \cdots + a_kx_k = y)$ and $G^*([n], a_1x_1 + \cdots + a_kx_k = y)$ games in Section 7, but how the fractional power $1/\ell$ interacts with the coefficients $a_1, \ldots, a_k$ is yet unknown.

Problem 2. What is the smallest integer n such that Maker wins the $G([n], a_1x_1^{1/\ell} + \cdots + a_kx_k^{1/\ell} = y^{1/\ell})$ game for $\ell \in \mathbb{Z}\backslash\{-1,0,1\}$? And what is the smallest integer n such that Maker wins the $G^*([n], a_1x_1^{1/\ell} + \cdots + a_kx_k^{1/\ell} = y^{1/\ell})$ game for $\ell \in \mathbb{Z}\backslash\{-1,0,1\}$?

Another direction is to study Rado games for the equation

$$x_1^{\ell} + \cdots + x_k^{\ell} = y^{\ell}, \tag{7}$$

where $\ell \in \mathbb{Z}\backslash\{-1,0,1\}$ and $k \in \mathbb{N}\backslash\{1\}$. In 2016, Heule, Kullmann, and Marek [15] verified that if $\{1, 2, \ldots, 7825\}$ is partitioned into two classes, then one of them contains a solution to Equation (7) with $k = \ell = 2$ and that there exists a partition of $\{1, 2, \ldots, 7824\}$ into two classes so that neither contains a solution to Equation (7) with $k = \ell = 2$. It is easy to see that if $a_1, a_2, b \in \mathbb{N}$ with $a_1^2 + a_2^2 = b^2$, then $a_1 \neq a_2$. So the result of Heule, Kullmann, and Marek implies that Maker wins both the $G([7825], x_1^2 + x_2^2 = y^2)$ game and the $G^*([7825], x_1^2 + x_2^2 = y^2)$ game. It would be interesting to see if Maker can do better.

Problem 3. Does there exist $n < 7825$ such that Maker wins the $G^*([n], x_1^2 + x_2^2 = y^2)$ game?

The situation for Maker is more complicated when $\ell \geq 3$. By Fermat's last theorem [25], for all $n, \ell \in \mathbb{N}$ with $\ell \geq 3$, Breaker wins both the $G([n], x_1^\ell + x_2^\ell = y^\ell)$ game and the $G^*([n], x_1^\ell + x_2^\ell = y^\ell)$ for $\ell \geq 3$. By homogeneity, Breaker also wins the $G([n], x_1^\ell + x_2^\ell = y^\ell)$ game and the $G^*([n], x_1^\ell + x_2^\ell = y^\ell)$ game for all $n \in \mathbb{N}$ and $\ell \leq -3$. Hence, in order to study Rado games for Equation (7), one needs extra conditions on k and ℓ to make sure there are solutions to Equation (7) in $\mathbb{N}$. Recently, Chow, Lindqvist, and Prendiville [8] proved that, for all $\ell \in \mathbb{N}$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, if we partition of $\mathbb{N}$ into two classes, then one of them contains a solution to Equation (7) with $x_1, \ldots, x_k$ not necessarily distinct. By the result of Brown and Rödl [6] described in Section 1, the same result holds for $\ell \in \{-1, -2, \ldots\}$ as well. For example, if $|\ell| = 2$, then $k = 4$ suffices; and if $|\ell| = 3$, then $k = 7$ is enough.

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