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A UNIFIED APPROACH TO qMZVS

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Abstract

We give a self-contained introduction to q-analogs of multiple zeta values (qMZVs) occurring in theoretical physics. For this, we consider the most common models of qMZVs in a unified setup going back to Bachmann and Kühn. Furthermore, we consider a related quasi-shuffle product and generating series for each model. As another unified approach to qMZVs, we introduce the concept of marked partitions.

1. Introduction

The *multiple zeta value* (MZV) of an *admissible index* $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$ (meaning that $k_1 \geq 2$, $r \in \mathbb{N}_0$) is

$$\zeta(\mathbf{k}) := \zeta(k_1, \ldots, k_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{m_1^{k_1}} \cdots \frac{1}{m_r^{k_r}},$$

where $\zeta(\emptyset) := 1$ for $r = 0$. We say that $\mathrm{wt}(\mathbf{k}) := k_1 + \cdots + k_r$ is the *weight* and $\mathrm{depth}(\mathbf{k}) := r$ is the *depth* of $\mathbf{k}$. The well-definedness of these terms follows using standard arguments from calculus.

MZVs can be represented by iterated (Kontsevich) integrals:

$$\zeta(k_1, \ldots, k_r) = \int\limits_{1 > t_1 > \cdots > t_k > 0} \omega_1(t_1) \ldots \omega_k(t_k),$$

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where $k := k_1 + \cdots + k_r$ and

$$\omega_i(t) := \begin{cases} \frac{dt}{1-t} & \text{if } i \in \{k_1, k_1 + k_2, \ldots, k_1 + \cdots + k_r\}, \\ \frac{dt}{t} & \text{otherwise.} \end{cases}$$

There are two representations of products of MZVs as rational weighted sums of MZVs. The stuffle product comes from the definition as iterated sums, and the shuffle product from Kontsevich's iterated integrals (the definitions of both products are given below). In particular, by considering products of MZVs, we get $\mathbb{Q}$-linear relations among MZVs, so-called double shuffle relations. After some regularization, conjecturally, these are all $\mathbb{Q}$-linear relations among MZVs, emphasizing the importance of stuffle and shuffle products.

Considering MZVs algebraically, one needs *quasi-shuffle algebras* (introduced by Hoffman [18]). For our purposes, these are $\mathbb{Q}$-algebras $\mathbb{Q}\langle A\rangle$ with A a finite set, $\diamond$ an associative and commutative product on $\mathbb{Q}A$, and where the product is a $\mathbb{Q}$-bilinear map $*_\diamond$ such that

$$\begin{aligned} &\mathbf{1} *_\diamond w = w *_\diamond \mathbf{1} := w, \\ &au *_\diamond bv := a(u *_\diamond bv) + b(au *_\diamond v) + (a \diamond b)(u *_\diamond v) \end{aligned}$$

for any $a, b \in \mathbb{Q}A$, and $u, v, w \in \mathbb{Q}\langle A\rangle$. Often, elements in A are called *letters*, and monoids in $\mathbb{Q}\langle A\rangle$ are called *words*.

Each product representation of MZVs can be described as an algebra homomorphism from a quasi-shuffle algebra to $(\mathbb{R}, \cdot)$. For this, define the free non-commutative algebra of two letters, $\mathfrak{h} := \mathbb{Q}\langle x_0, x_1\rangle$. Consider $\mathfrak{h}^0 := \mathbb{Q}\mathbf{1} \oplus x_0\mathfrak{h}x_1$ and the evaluation map $\zeta : \mathfrak{h}^0 \to \mathbb{R}$ via $\mathbf{1} \mapsto 1$,

$$z_{k_1} \cdots z_{k_r} \longmapsto \zeta(k_1, \ldots, k_r),$$

and $\mathbb{Q}$-linear extension with the abbreviation $z_k := x_0^{k-1}x_1$.

Consider two particular quasi-shuffle products. First, we consider the one on $\mathbb{Q}\langle A\rangle$, where $A = \{z_k = x_0^{k-1}x_1 \,|\, k \geq 1\}$ and the diamond product is given by $z_m \diamond z_n := z_{m+n}$. The induced quasi-shuffle product $* := *_\diamond$ on $\mathbb{Q}\langle A\rangle$ is called the *stuffle product* (see page 14 of [5]) and is closed under restriction to $\mathfrak{h}^0$.

Second, we consider the *shuffle product* (see page 14 of [5]). This is the quasi-shuffle product $⧢$ on $\mathfrak{h}$ (we choose $A = \{x_0, x_1\}$ so that $\mathbb{Q}\langle A\rangle = \mathfrak{h}$) which is induced by the diamond constant 0. It is also closed under restriction to $\mathfrak{h}^0$.

By [19, Proposition 1], both $\zeta : (\mathfrak{h}^0, *) \to (\mathbb{R}, \cdot)$ and $\zeta : (\mathfrak{h}^0, ⧢) \to (\mathbb{R}, \cdot)$ are algebra homomorphisms. The first embodies the product of MZVs obtained from iterated sums, and the second is obtained from iterated integrals.

To understand the algebraic structure of MZVs, it is often helpful to consider q-analogs (qMZVs). Furthermore, thinking of qMZVs as holomorphic functions in the unit disc gives connections to quasi-modular forms. Since several models of qMZVs are often introduced in different ways, in this article we give a unified approach to all of these models. For this, we present "general" qMZVs, and we will see that every model we consider has the specific shape of these general qMZVs.

Definition 1 (qMZV, [8, Equation (1)]). (i) Define for $r \geq 0$, $k_1, \ldots, k_r \geq 0$, and polynomials $Q_1 \in X\mathbb{Q}[X]$, $Q_2, \ldots, Q_r \in \mathbb{Q}[X]$ with $\deg(Q_j) \leq k_j$ for all j, the *qMZV*

$$\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{Q_1(q^{m_1})}{(1-q^{m_1})^{k_1}} \cdots \frac{Q_r(q^{m_r})}{(1-q^{m_r})^{k_r}} \in \mathbb{Q}[\![q]\!],$$

with $\zeta_q(\emptyset, \emptyset) := 1$, where q is a formal variable.
(ii) Define $\mathcal{Z}_q$ as

$$\langle \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \,|\, r \geq 0,\, k_j \geq 0,\, Q_1 \in X\mathbb{Q}[X],\, \deg(Q_j) \leq k_j \rangle_{\mathbb{Q}}.$$

We describe the subspace of $\mathcal{Z}_q$ spanned by every model and give a shuffle product that the model satisfies. Since it is sometimes useful, we provide distinguished translations of several models into others. We will demonstrate the translations directly and on generating series. One of the applications is proving that the Schlesinger–Zudilin duality and the partitions relation are equivalent (see [12, Theorem 3.22]). These subspaces are then considered in more detail in Section 3, where we refer to Figure 3 for a brief overview.

Section 4 yields a purely combinatorial view to qMZVs connecting qMZVs with certain partitions, called *marked partitions*. The main result is the following combinatorial identity, which is obtained from BZ duality (Theorem 7), a relation of qMZVs. A proof can be deduced from Section 4.3.

Theorem 1 ([12, Theorem 4.18]). *For all $r \geq 1$, $k_1, \ldots, k_r, d_1, \ldots, d_r \geq 1$, and $N \geq 1$, we have*

$$\sum_{\substack{m_1 > n_1 > \cdots > n_{d_1-1} > m_2 > \cdots > m_r > \cdots > n_{d_1+\cdots+d_r-r} > 0 \\ j_1, i_1, \ldots, i_{d_1-1}, j_2, \ldots, i_{d_1+\cdots+d_r-r} \geq 0 \\ m_1 j_1 + \cdots + m_r j_r + n_1 i_1 + \cdots + n_{d_1+\cdots+d_r-r} i_{d_1+\cdots+d_r-r} = N}} \binom{j_1}{k_1} \cdots \binom{j_r}{k_r}$$
$$= \sum_{\substack{m_1 > n_1 > \cdots > n_{k_r-1} > m_2 > \cdots > m_r > \cdots > n_{k_r+\cdots+k_1-r} > 0 \\ j_1, i_1, \ldots, i_{k_r-1}, j_2, \ldots, i_{k_r+\cdots+k_1-r} \geq 0 \\ m_1 j_1 + \cdots + m_r j_r + n_1 i_1 + \cdots + n_{k_r+\cdots+k_1-r} i_{k_r+\cdots+k_1-r} = N}} \binom{j_1}{d_r} \cdots \binom{j_r}{d_1}.$$

2. Models of qMZVs

We begin by considering general modified q-analogs of MZVs (Definition 1) and their connection to MZVs. Furthermore, we first see $\mathcal{Z}_q$, the $\mathbb{Q}$-algebra of qMZVs. A natural question is which elements generate this algebra, which leads to different models of modified qMZVs. Every model of qMZVs contains at least one algebraic aspect of MZVs: Schlesinger–Zudilin's model inherits the stuffle product, and Bradley–Zhao's model the duality of MZVs. Also important is Bachmann's model since it gives a deep and direct connection to quasi-modular forms that play an essential role in the theory of MZVs as Gangl, Kaneko, and Zagier [17], as well as Broadhurst and Kreimer [13], have shown. For more details about the various models, we refer to the original works [2, 11, 22, 23, 25, 30, 33, 36]. Also, in [34] the author gives an overview of the models and their history.

In general, a q-analog of an object is a modified object in an additional variable q (often a series in (complex) q with $|q| < 1$) that returns the original one in the limit as q approaches 1, taken on the real axis from the left. For example, the standard q-analog of a natural number n is

$$[n]_q := \frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}.$$

Modified q-analogs of MZVs are q-series that return a multiple zeta value if we multiply the q-series first with a specific power of $(1-q)$ and then take the limit as q approaches 1. We consider only modified qMZVs and avoid the word "modified" in the following.

Remark 1. (i) The condition $Q_1 \in X\mathbb{Q}[X]$ in Definition 1 is necessary for well-definedness.

(ii) Note that $\mathcal{Z}_q$ does not necessarily contain all modified q-analogs of MZVs. For example, it still needs to be determined whether the modified qMZVs introduced by Shen and Qin [27] are in $\mathcal{Z}_q$.

We use notation from [8], where the authors introduce important subspaces of $\mathcal{Z}_q$.

Definition 2. For $d \geq 0$, define

$$\mathcal{Z}_{q,d} := \langle \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \in \mathcal{Z}_q | \, r \geq 0, \; k_j \geq 1, \deg(Q_j) \leq k_j - d \rangle_{\mathbb{Q}},$$
$$\mathcal{Z}^{\circ}_{q,d} := \langle \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) \in \mathcal{Z}_{q,d} | \, r \geq 0, \; k_j \geq 1, Q_j \in X\mathbb{Q}[X] \rangle_{\mathbb{Q}}.$$

We will always denote $\mathcal{Z}^{\circ}_q := \mathcal{Z}^{\circ}_{q,0}$.

Remark 2. Naively, we could think of $k_1 + \cdots + k_r$ as the "weight" of the qMZV $\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r)$ in accordance with the definition of weight for MZVs. But this is not well-defined since, for example, we have

$$\zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) = \zeta_q(k_1 + 1, \ldots, k_r; (1 - X)Q_1, Q_2 \ldots, Q_r).$$

Hence, we need another notion of weight. We will consider such a notion for bi-brackets (Definition 11).

Remark 3. (i) The spaces $\mathcal{Z}_q$ and $\mathcal{Z}_q^\circ$ are closed under the operator $q\frac{d}{dq}$ (see [4, Proposition 4.2], or [7, Proposition 3.14]).

(ii) The spaces $\mathcal{Z}_{q,1}$ and $\mathcal{Z}_{q,1}^\circ$ are conjecturally closed under $q\frac{d}{dq}$ (see [4, Conjecture 4.3], or [23, Conjecture 1]).

Proposition 1 ([12, Remark 2.2])**.** (i) *Every qMZV converges for complex q with $|q| < 1$ and is a holomorphic function in the upper half plane via $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H}$.*

(ii) *For $k_1 \geq 2$, $k_2, \ldots, k_r \geq 1$, the limit*

$$\lim_{q \to 1^-} (1 - q)^{k_1 + \cdots + k_r} \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r) = \zeta_q(k_1, \ldots, k_r) \prod_{j=1}^{r} Q_j(1)$$

exists and is obtained by interchanging limit and summation due to absolute convergence on $[0, 1)$*.*

Often, we will talk about $\mathcal{Z}_q$ as an algebra. We justify this in the following as we will see that we can give $\mathcal{Z}_q$ a structure such that it becomes a quasi-shuffle algebra.

Definition 3. Consider the alphabet

$$A_Z := \left\{ \binom{k}{Q} \mid Q \in \mathbb{Q}[X],\, k \in \mathbb{N},\, \deg(Q) \leq k \right\}.$$

Define the product $\diamond : \mathbb{Q}A_Z \times \mathbb{Q}A_Z \to \mathbb{Q}A_Z$, induced by

$$\left(\binom{k_1}{Q_1}, \binom{k_2}{Q_2} \right) \longmapsto \binom{k_1 + k_2}{Q_1 \cdot Q_2},$$

via $\mathbb{Q}$-bilinearity, $w \diamond \mathbf{1} := \mathbf{1} \diamond w := w$ for all $w \in \mathbb{Q}A_Z$.

Define $A_Z^\circ := \left\{ \binom{k}{Q} : Q \in X\mathbb{Q}[X],\, k \in \mathbb{N},\, \deg(Q) \leq k \right\}$ and consider the algebra $M := A_Z^\circ \mathbb{Q} \langle A_Z \rangle \oplus \mathbb{Q}\mathbf{1}$. Let $*$ be the induced quasi-shuffle product on M. It turns out that ζ_q becomes an algebra homomorphism.

Proposition 2. *The evaluation map* $\zeta_q : M \to \mathcal{Z}_q$*,*

$$\binom{k_1}{Q_1} \cdots \binom{k_r}{Q_r} \longmapsto \zeta_q(k_1, \ldots, k_r; Q_1, \ldots, Q_r),$$

extended to M *by* $\mathbb{Q}$*-linearity, is an algebra homomorphism, i.e., for all* $u, v \in M$ *we have*

$$\zeta_q(u * v) = \zeta_q(u)\zeta_q(v).$$

Proof. The proposition follows from the multiplication of qMZVs, represented as iterated sums. □

Remark 4 ([8])**.** The subspaces $\mathcal{Z}_{q,d}$, $\mathcal{Z}^{\circ}_{q,d} \subseteq \mathcal{Z}_q$ are, for all $d \geq 0$, subalgebras of $\mathcal{Z}_q$ by restricting $*$ to the corresponding subspace.

The notion of some free non-commutative algebras is needed to describe quasi-shuffle products in different models of qMZVs.

Definition 4. Define the free non-commutative algebra in the variables x_0 and x_1 (respectively, p and y),

$$\mathfrak{h} := \mathbb{Q}\left\langle x_0, x_1 \right\rangle, \quad \mathfrak{K} := \mathbb{Q}\left\langle p, y \right\rangle.$$

Furthermore, define the subalgebras ($\mathbf{1}$ is the unit in the following)

$$\begin{aligned} \mathfrak{h}^0 &:= x_0 \mathfrak{h} x_1 \oplus \mathbb{Q}\mathbf{1}, & \mathfrak{h}^1 &:= \mathfrak{h} x_1 \oplus \mathbb{Q}\mathbf{1}, \\ \mathfrak{K}^1 &:= p\mathfrak{K}y \oplus \mathbb{Q}\mathbf{1}, & \mathfrak{K}^3 &:= p\mathbb{Q}\left\langle p, py \right\rangle py \oplus \mathbb{Q}\mathbf{1}. \end{aligned}$$

Monomials in the *letters* x_0 and x_1 (respectively, p and y) are called *words*; $\mathbf{1}$ is the *empty word*.

2.1. Schlesinger–Zudilin Model

One of the most natural questions is how bases, or at least generating systems of the $\mathbb{Q}$-vector space $\mathcal{Z}_q$, appear and whether there are interesting subspaces we should consider. An example of such a generating system (cf. Proposition 3) is

$$\left\{ \zeta_q\left(k_1, \ldots, k_r; X^{k_1}, \ldots, X^{k_r}\right) \,\middle|\, r \geq 0,\, k_1 \geq 1,\, k_2, \ldots, k_r \geq 0 \right\}.$$

These generators are named *Schlesinger–Zudilin qMZVs*, introduced independently by Schlesinger [25] and Zudilin [35].

Definition 5 (Schlesinger–Zudilin qMZVs)**.**

(i) An index $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}_0^r$ is *SZ admissible* if $r \geq 0$ and $\mathbf{k} = \emptyset$ or $k_1 \geq 1$.

(ii) Define for every SZ admissible index $\mathbf{k}$ the *Schlesinger–Zudilin qMZV as* $\zeta_q^{\mathrm{SZ}}(\emptyset) := 1$ *and, for* $r \geq 1$,

$$\begin{aligned}\zeta_q^{\mathrm{SZ}}(\mathbf{k}) :=& \zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r) := \zeta_q\left(k_1, \ldots, k_r; X^{k_1}, \ldots, X^{k_r}\right) \\ =& \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}}.\end{aligned}$$

We introduced an extended version due to Ebrahimi-Fard, Manchon, and Singer (cf. [15]). In the original model, due to Schlesinger and Zudilin, only indices with $k_i \geq 1$ were considered.

Remark 5. (i) If one of the indices is 0 in an SZ qMZV, $k_j = 0$ for some j, then the summand is independent of m_j. Hence, it is often useful to distinguish between zero and non-zero indices.

(ii) An index $\mathbf{k}$ is SZ admissible if and only if $\mathbf{k} + \mathbf{1}$ is admissible [2].

(iii) The name of the SZ model is attributed not only to Schlesinger [25], although his work was two years before Zudilin's [35], since Schlesinger considered

$$\zeta_q^{\mathrm{SZ}'}(k_1, \ldots, k_r) := \sum_{m_1 > \cdots > m_r > 0} \frac{1}{(1-q^{m_1})^{k_1} \cdots (1-q^{m_r})^{k_r}}$$

with $|q| > 1$ instead of $\zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r)$ (with $|q| < 1$). The latter is nowadays the usual definition which is due to Zudilin. On closer inspection, one sees that $\zeta_q^{\mathrm{SZ}'}$ and ζ_q^{SZ} almost coincide. Namely, one has

$$\zeta_{q^{-1}}^{\mathrm{SZ}'}(k_1, \ldots, k_r) = (-1)^{k_1 + \cdots + k_r} \zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r).$$

Further details of the history of SZ qMZVs can be found, e.g., in [33].

(iv) For some applications such as translation or duality in the OOZ model (Theorem 17), it is useful to have the index in the defining sum of SZ qMZVs not strictly ordered. This leads to the definition of SZ-star qMZVs, particular finite sums of SZ qMZVs, for SZ admissible $\mathbf{k}$ defined as

$$\zeta_q^{\mathrm{SZ},\star}(\mathbf{k}) := \zeta_q^{\mathrm{SZ},\star}(k_1, \ldots, k_r) := \sum_{m_1 \geq \cdots \geq m_r > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}}.$$

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[2] $\mathbf{k} + \mathbf{1}$ is the index $\mathbf{k}$ with every argument increased by 1.

Proposition 3. *The SZ model is closed under* $q\frac{d}{dq}$*, and the SZ qMZVs span* $\mathcal{Z}_q$*, i.e.,*

$$\mathcal{Z}_q = \left\langle \zeta_q^{\mathrm{SZ}}(k_1,\ldots,k_r) \,\middle|\, r \geq 0,\, k_1 \geq 1,\, k_i \geq 0 \right\rangle_{\mathbb{Q}}.$$

Proof. The proof is obtained from the fact that every expression $\frac{X^n}{(1-X)^s}$ for $0 \leq n \leq s$ is a finite $\mathbb{Q}$-linear combination of terms $\frac{X^k}{(1-X)^k}$ for $k \geq 0$. Specifically, this applies for $s \in \mathbb{N}_0$ and $0 \leq n \leq s$,

$$\frac{X^n}{(1-X)^s} = \sum_{p=n}^{s} \binom{s-n}{p-n} \frac{X^p}{(1-X)^p}. \tag{2.1}$$

By Remark 3(i), the SZ model is, in particular, closed under $q\frac{d}{dq}$. □

SZ qMZVs satisfy a quasi-shuffle product similar to the stuffle product of MZVs and some duality relation. Combined, they imply the shuffle product of MZVs (cf. [15] and [28]).

Definition 6 (SZ stuffle product)**.** (i) Set $u_k := p^k y \in \mathfrak{R}$ for all $k \in \mathbb{N}_0$.

(ii) Consider on $\mathfrak{R}$ the *SZ stuffle product* $*_{\mathrm{SZ}} : \mathfrak{R} \times \mathfrak{R} \to \mathfrak{R}$, recursively given by distributivity and

(i) $\mathbf{1} *_{\mathrm{SZ}} w = w *_{\mathrm{SZ}} \mathbf{1} := w$,

(ii) $u_s v *_{\mathrm{SZ}} u_t w := u_s(v *_{\mathrm{SZ}} u_t v) + u_t(u_s v *_{\mathrm{SZ}} w) + u_{s+t}(v *_{\mathrm{SZ}} w)$

for all words $v, w \in \mathfrak{R}$, and $s, t \in \mathbb{N}_0$.

Note that $\mathfrak{R}^1$ is generated by words starting in an u_k, $k \geq 1$. Hence, $\mathfrak{R}^1$ is closed under $*_{\mathrm{SZ}}$.

Definition 7. Identifying $u_{k_1} \ldots u_{k_r} \in \mathfrak{R}^1$ with $(k_1, \ldots, k_r)$, we define the map

$$\zeta_q^{\mathrm{SZ}} : \mathfrak{R}^1 \longrightarrow \mathbb{Q}[\![q]\!], \qquad u_{k_1} \cdots u_{k_r} \longmapsto \zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r)$$

and extend ζ_q^{SZ} to $\mathfrak{R}^1$ by $\mathbb{Q}$-linearity and $\mathbf{1} \mapsto 1$.

The following states that ζ_q^{SZ} is an algebra homomorphism.

Theorem 2. *The map* ζ_q^{SZ} *is an algebra homomorphism on* $(\mathfrak{R}^1, *_{\mathrm{SZ}})$*. In particular, for all* $v, w \in \mathfrak{R}^1$*, we have*

$$\zeta_q^{\mathrm{SZ}}(v)\zeta_q^{\mathrm{SZ}}(w) = \zeta_q^{\mathrm{SZ}}(v *_{\mathrm{SZ}} w).$$

Proof. The statement follows from the definition of SZ qMZVs as iterated sums. ☐

We can consider SZ qMZVs in another algebraic way.

Definition 8. (a) Define the *SZ qshuffle product* $ш_{\mathrm{SZ}} : \mathfrak{K} \times \mathfrak{K} \to \mathfrak{K}$ recursively via

(i) $\mathbf{1} ш_{\mathrm{SZ}} w = w ш_{\mathrm{SZ}} \mathbf{1} := w$,

(ii) $yu ш_{\mathrm{SZ}} v = u ш_{\mathrm{SZ}} yv := y(u ш_{\mathrm{SZ}} v)$,

(iii) $pu ш_{\mathrm{SZ}} pv := p(u ш_{\mathrm{SZ}} pv) + p(pu ш_{\mathrm{SZ}} v) + p(u ш_{\mathrm{SZ}} v)$,

distributivity, and $\mathbb{Q}$-bilinearity for all $u, v, w \in \mathfrak{K}$.

(b) We identify an SZ admissible index $\mathbf{k} = (k_1, \ldots, k_r)$ with the word $p^{k_1}y \ldots p^{k_r}y \in \mathfrak{K}^1$. Then we can define ζ_q^{SZ} as the more general map

$$\zeta_q^{\mathrm{SZ}} : \mathfrak{K}^1 \longrightarrow \mathcal{Z}_q, \qquad p^{k_1}y \cdots p^{k_r}y \longmapsto \zeta_q^{\mathrm{SZ}}(k_1, \ldots, k_r),$$

extended to $\mathfrak{K}^1$ by $\mathbb{Q}$-linearity and mapping $\mathbf{1} \mapsto 1$.

Singer proved that ζ_q^{SZ} is an algebra homomorphism on $(\mathfrak{K}^1, ш_{\mathrm{SZ}})$.

Theorem 3 ([28, Theorem 5]). *The map ζ_q^{SZ} is an algebra homomorphism on $(\mathfrak{K}^1, ш_{\mathrm{SZ}})$, i.e., for all words $u, v \in \mathfrak{K}^1$ we have*

$$\zeta_q^{\mathrm{SZ}}(u)\zeta_q^{\mathrm{SZ}}(v) = \zeta_q^{\mathrm{SZ}}(u ш_{\mathrm{SZ}} v).$$ ☐

Often - as for an elegant proof of SZ duality (Theorem 5) - it is helpful to consider the generating series of, e.g., SZ qMZVs.

Theorem 4 ([12, Theorem 2.9]). *For every $r \geq 1$, define*

$$\mathfrak{s}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} := \sum_{\substack{k_1, \ldots, k_r \geq 1 \\ d_1, \ldots, d_r \geq 1}} \zeta_q^{\mathrm{SZ}}\left(k_1, \{0\}^{d_1-1}, \ldots, k_r, \{0\}^{d_r-1}\right) \prod_{j=1}^{r} X_j^{k_j-1} Y_j^{d_j-1}.$$

Then, for every $r \geq 1$, with $m_{r+1} := 0$ we have

$$\mathfrak{s}\binom{X_1, \ldots, X_r}{Y_1, \ldots, Y_r} = \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r \geq 1}} \prod_{j=1}^{r} (1 + X_j)^{n_j-1}(1 + Y_j)^{m_j-m_{j+1}-1} q^{m_j n_j}.$$

Proof. With the combinatorial identity for all $M_1, M_2, \ell \in \mathbb{N}_0$,

$$\#\{n_1, \dots, n_\ell \in \mathbb{N} \,\big|\, M_1 > n_1 > \cdots > n_\ell > M_2\} = \binom{M_1 - M_2 - 1}{\ell},$$

geometric series expansion, and the binomial theorem, we get $(m_{r+1} := 0)$

$$\begin{aligned}
\mathfrak{s}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} &= \sum_{\substack{k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 1}} \sum_{\substack{m_1 > n_1 > \cdots > n_{d_1-1} > m_2 \\ > \cdots > 0}} \prod_{j=1}^{r} \frac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}} X_j^{k_j-1} Y_j^{d_j-1} \\
&= \sum_{\substack{k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 1}} \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \tbinom{m_j - m_{j+1} - 1}{d_j - 1} \tfrac{q^{m_j k_j}}{(1-q^{m_j})^{k_j}} X_j^{k_j-1} Y_j^{d_j-1} \\
&= \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1,\dots,n_r > 0}} \prod_{j=1}^{r} \left(\sum_{\substack{k_j \geq 0 \\ d_j \geq 0}} \tbinom{m_j - m_{j+1} - 1}{d_j} \tbinom{n_j - 1}{k_j} X_j^{k_j} Y_j^{d_j} q^{m_j n_j} \right) \\
&= \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1,\dots,n_r > 0}} \prod_{j=1}^{r} (1 + X_j)^{n_j - 1} (1 + Y_j)^{m_j - m_{j+1} - 1} q^{m_j n_j}. \qquad \square
\end{aligned}$$

SZ qMZVs satisfy a duality relation similar to the one of MZVs, which is, together with some related statements, why they are interesting objects.

Theorem 5 (SZ Duality [34, Theorem 8.3])**.** *Let* $\tilde{\tau} : \mathfrak{K} \to \mathfrak{K}$ *be the anti-automorphism with respect to concatenation, induced by* $\tilde{\tau}(p) := y$, $\tilde{\tau}(y) := p$. *On* $\mathfrak{K}^1$ *we have*

$$\zeta_q^{\mathrm{SZ}} \circ \tilde{\tau} = \zeta_q^{\mathrm{SZ}}.$$

Equivalently, for every SZ admissible $\mathbf{k} = \big(k_1, \{0\}^{d_1-1}, \dots, k_r, \{0\}^{d_r-1}\big)$, *we have*

$$\zeta_q^{\mathrm{SZ}}(\mathbf{k}) = \zeta_q^{\mathrm{SZ}}(\mathbf{k}^\dagger),$$

where $\mathbf{k}^\dagger := \big(d_r, \{0\}^{k_r-1}, \dots, d_1, \{0\}^{k_1-1}\big)$ *is the SZ dual index of* $\mathbf{k}$.

Proof. The theorem follows, as shown in [12], immediately from the identity

$$\mathfrak{s}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \mathfrak{s}\binom{Y_r, \dots, Y_1}{X_r, \dots, X_1}. \qquad \square$$

Remark 6. We saw that the SZ model satisfies a q-analog of the stuffle product of MZVs and some duality relation. An application of the SZ model is that the SZ stuffle product induces, together with SZ duality, the shuffle product of MZVs (see [28, Theorem 9], or [12, Theorems 3.46 and 3.52]).

2.2. Bradley–Zhao Model

In depth one, the BZ model of qMZVs was first considered by Kaneko, Kurokawa, and Wakayama [21]. The general one was then introduced by Zhao [33] and independently by Bradley [11]. BZ qMZVs satisfy the same duality as MZVs which is why this model plays an essential role in the context of MZVs and qMZVs.

Definition 9 (Bradley–Zhao qMZVs)**.** We define $\zeta_q^{\mathrm{BZ}}(\emptyset) := 1$ and for every *admissible index* $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r$, i.e., $k_1 \geq 2$, with $r \geq 1$, we define the *Bradley–Zhao qMZV* as

$$\begin{aligned}\zeta_q^{\mathrm{BZ}}(\mathbf{k}) := \zeta_q^{\mathrm{BZ}}(k_1, \dots, k_r) &:= \zeta_q\left(k_1, \dots, k_r; X^{k_1-1}, \dots, X^{k_r-1}\right) \\ &= \sum_{m_1 > \cdots > m_r > 0} \frac{q^{m_1(k_1-1)}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r(k_r-1)}}{(1-q^{m_r})^{k_r}}.\end{aligned}$$

In contrast to the SZ model, BZ qMZVs span a proper subspace of $\mathcal{Z}_q$.

Proposition 4. *The span of the BZ model is $\mathcal{Z}_{q,1}$,*

$$\mathcal{Z}_{q,1} = \left\langle \zeta_q^{\mathrm{BZ}}(k_1, \dots, k_r) \,\middle|\, r \geq 0,\ k_1 \geq 2,\ k_i \geq 1 \right\rangle_{\mathbb{Q}}.$$

Proof. Every BZ qMZV is by definition an element of $\mathcal{Z}_{q,1}$. Also, every element of $\mathcal{Z}_{q,1}$ can be written as a rational linear combination following from the identity

$$\frac{X^s}{(1-X)^n} = \frac{X^s}{(1-X)^{s+1}}\left(1 + \frac{X}{1-X}\right)^{n-s+1},$$

holding true for all $0 \leq s < n$. □

Note that $\mathcal{Z}_{q,1}$ is a proper subspace of $\mathcal{Z}_q$ since $\zeta_q(1; X) \in \mathcal{Z}_q$, e.g., can not be written in terms of BZ qMZVs. One can prove this fact with arguments similar to [7, Theorem 2.14 (ii)].

BZ qMZVs satisfy a quasi-shuffle product, in analogy to the stuffle product of MZVs, since the multiplication of iterated sums induces it.

Definition 10. (i) Consider ζ_q^{BZ} as map $\zeta_q^{\mathrm{BZ}} : \mathfrak{h}^0 \to \mathcal{Z}_q$ via $\mathbb{Q}$-linearity, $\mathbf{1} \mapsto 1$ and

$$z_{k_1} \cdots z_{k_r} \longmapsto \zeta_q^{\mathrm{BZ}}(k_1, \dots, k_r).$$

(ii) Define on $\mathbb{Q}\{z_k : k \in \mathbb{N}\}$ the commutative and associative product $\diamond_{\mathrm{BZ}}$ via

$$z_{k_1} \diamond_{\mathrm{BZ}} z_{k_2} := z_{k_1+k_2} + z_{k_1+k_2-1}$$

for all $k_1, k_2 \geq 1$, and $\mathbf{1} \diamond_{\mathrm{BZ}} w := w \diamond_{\mathrm{BZ}} \mathbf{1} := w$ for all $w \in \mathfrak{h}^0$. Let be $*_{\mathrm{BZ}}$ the induced quasi-shuffle product on $\mathfrak{h}^1$.

Notice that $\mathfrak{h}^0 \subset \mathfrak{h}^1$ is closed under $*_{\mathrm{BZ}}$. By considering products of iterated sums, one obtains the following result.

Proposition 5. *On $(\mathfrak{h}^0, *_{\mathrm{BZ}})$, ζ_q^{BZ} is an algebra homomorphism, i.e., for all $u, v \in \mathfrak{h}^0$ we have*

$$\zeta_q^{\mathrm{BZ}}(u *_{\mathrm{BZ}} v) = \zeta_q^{\mathrm{BZ}}(u)\zeta_q^{\mathrm{BZ}}(v).$$

For introducing a generating series of BZ qMZVs, define for all $\ell \in \mathbb{N}$ (both rows have ℓ entries)

$$f_\ell(X, Y) := \binom{X, 0, \dots, 0}{Y, Y, \dots, Y}.$$

We identify $(f_{\ell_1}, \dots, f_{\ell_r})$, $r, \ell_j \in \mathbb{N}$, with the row-wise concatenation of $f_{\ell_1}, \dots, f_{\ell_r}$.

Theorem 6 ([12, Theorem 2.13])**.** *For $r \geq 1$, define the generating series of BZ qMZVs,*

$$\mathfrak{b}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{\substack{k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 1}} \zeta_q^{\mathrm{BZ}}\left(k_1 + 1, \{1\}^{d_1-1}, \dots, k_r + 1, \{1\}^{d_r-1}\right) \prod_{j=1}^{r} X_j^{k_j-1} Y_j^{d_j-1}.$$

Then, for every $r \geq 1$, we have

$$\mathfrak{b}\binom{X_1, \dots, X_r}{Y_1, \dots, Y_r} = \sum_{\substack{\ell_1,\dots,\ell_r \geq 1 \\ \delta_1,\dots,\delta_r \in \{0,1\}}} (-1)^{r-(\delta_1+\dots+\delta_r)} \mathfrak{s}\left(f_{\ell_1}(\delta_1 X_1, Y_1), \dots, f_{\ell_r}(\delta_r X_r, Y_r)\right) \times \prod_{j=1}^{r} (1 + \delta_j X_j) Y_j^{\ell_j - 1}.$$

Remark 7. Considering Theorem 6 modulo terms not divisible by $\prod_{j=1}^{r} X_jY_j$ yields, for all $r \geq 1$,

$$\mathfrak{b}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} \equiv \sum_{\ell_1,\dots,\ell_r\geq 1} \mathfrak{s}\left(f_{\ell_1}(X_1,Y_1),\dots,f_{\ell_r}(X_r,Y_r)\right)\prod_{j=1}^{r}(1+\delta_j X_j)Y_j^{\ell_j-1}.$$

One of the reasons why BZ qMZVs are of interest is that they satisfy the same duality as MZVs.

Theorem 7 (BZ duality, [11, Theorem 5])**.** *On* $\mathfrak{h}^0$ *we have* BZ *duality,*

$$\zeta_q^{\mathrm{BZ}} \circ \tau = \zeta_q^{\mathrm{BZ}}. \qquad \square$$

There is a larger class of relations, the q-Ohno relations. BZ duality is the special case $c = 0$.

Theorem 8 (q-Ohno relation, [24, Theorem 1])**.** *For every admissible* $\mathbf{k} = (k_1,\dots,k_r)$ *and* $c \in \mathbb{N}_0$*, we have*

$$\sum_{|\mathbf{c}|=c} \zeta_q^{\mathrm{BZ}}(\mathbf{k}+\mathbf{c}) = \sum_{|\mathbf{c}'|=c} \zeta_q^{\mathrm{BZ}}\left(\mathbf{k}^\vee + \mathbf{c}'\right),$$

where we sum over all $\mathbf{c} \in \mathbb{N}_0^r$*, respectively* $\mathbf{c}' \in \mathbb{N}_0^{r'}$ *(* $|\cdot|$ *deontes the sum of the entries), with* $r' = \operatorname{depth}(\mathbf{k}^\vee)$*, and with componentwise addition of indices.*

Remark 8. The indisputable advantage of this model is that it satisfies the same duality as MZVs. In particular, this follows from BZ duality by taking the limit as q approaches 1 after multiplication with $(1-q)^{k_1+\cdots+k_r}$.

2.3. Bi-brackets

Another important model of q-analogs are so-called brackets (introduced in Bachmann's master thesis [1], further investigated, e.g., in [7]) and their generalization, bi-brackets, introduced by Bachmann in his Ph.D. thesis [2].

The motivation for introducing bi-brackets came originally from examining the Fourier expansion of the Eisenstein series and their generalization, Multiple Eisenstein series, such as their derivatives (studied in [4]). From there, the original definition is justified.

Definition 11 ([4, Definition 2.1])**.** (i) For every $r \geq 0$, $k_1, \ldots, k_r \in \mathbb{N}$, and $d_1, \ldots, d_r \in \mathbb{N}_0$, the *bi-bracket* is

$$\begin{aligned} \mathrm{g}\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} &:= \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \prod_{j=1}^{r} \frac{m_j^{d_j}}{d_j!} \frac{n_j^{k_j - 1}}{(k_j - 1)!} q^{m_j n_j} \\ &= \sum_{m_1 > \cdots > m_r > 0} \frac{m_1^{d_1}}{d_1!} \cdots \frac{m_r^{d_r}}{d_r!} \frac{P_{k_1}(q^{m_1})}{(1 - q^{m_1})^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{(1 - q^{m_r})^{k_r}}, \end{aligned}$$

where P_k is the kth *Eulerian polynomial*,

$$\begin{aligned} P_k(X) :=& (1 - X)^k \sum_{n > 0} \frac{n^{k-1}}{(k-1)!} X^n \\ =& \frac{1}{(k-1)!} \sum_{n=1}^{k} \left(\sum_{j=0}^{n-1} (-1)^j \binom{k}{j} (n - j)^{k-1} \right) X^n. \end{aligned}$$

Additionally, we set $\mathrm{g}\binom{\emptyset}{\emptyset} := 1$ as usual. We define the *weight* $k_1 + \cdots + k_r + d_1 + \cdots + d_r$, and we call r the *depth* of the bi-bracket.

(ii) We define $\mathrm{g}(\emptyset) := 1$, and for any $r \geq 1$ and any $\mathbf{k} = (k_1, \ldots, k_r) \in \mathbb{N}^r$, we define the *bracket* of $\mathbf{k}$ as

$$\begin{aligned} \mathrm{g}(\mathbf{k}) := \mathrm{g}(k_1, \ldots, k_r) &:= \zeta_q(k_1, \ldots, k_r; P_{k_1}, \ldots, P_{k_r}) \\ &= \sum_{m_1 > \cdots > m_r > 0} \frac{P_{k_1}(q^{m_1})}{(1 - q^{m_1})^{k_1}} \cdots \frac{P_{k_r}(q^{m_r})}{(1 - q^{m_r})^{k_r}}. \end{aligned}$$

Remark 9. (i) The name *(bi-)bracket* comes from the original notation with brackets $[\ldots]$ instead of $\mathrm{g}(\ldots)$.

(ii) All brackets $\mathrm{g}(k_1, \ldots, k_r)$ are bi-brackets since $\mathrm{g}(k_1, \ldots, k_r) = \mathrm{g}\binom{k_1, \ldots, k_r}{0, \ldots, 0}$.

Bi-brackets generalize Eisenstein series since for even k, $\mathrm{g}\binom{k}{0}$ is the usual Eisenstein series G_k of weight k up to the constant term, i.e.,

$$G_k = -\frac{B_k}{2k!} + \mathrm{g}\binom{k}{0}.$$

Furthermore, for every $d > 0$, we have

$$\left(q \frac{d}{dq} \right)^d G_k = \frac{(k + d - 1)! d!}{(k-1)!} \mathrm{g}\binom{k + d}{d}.$$

The previous observation shows that the space of quasi-modular forms, $\mathbb{Q}[G_2, G_4, G_6]$, is a proper subspace of $\mathcal{Z}_q$. In this way, we get a connection to modular forms, which play an important role in the theory of MZVs as considered, e.g., in [17].

Bi-brackets and their structure are well known. We refer to [3, 4, 5, 7, 8, 36] for more details than in this section.

Theorem 9 ([8, Theorem 1 (i)]). *Bi-brackets span the space $\mathcal{Z}_q$, i.e.,*

$$\mathcal{Z}_q = \left\langle \mathrm{g}\begin{pmatrix} k_1, \dots, k_r \\ d_1, \dots, d_r \end{pmatrix} \,\middle|\, r \geq 0,\ k_i \geq 1,\ d_i \geq 0 \right\rangle_{\mathbb{Q}}.$$

Also, the algebra of bi-brackets can be viewed as a quasi-shuffle algebra (see Theorem 3.6, and the lines before, in [4]).

Definition 12 ([4]). Set $A_z^{bi} := \{z_{k,d} \,|\, k, d \in \mathbb{N}_0,\ k \geq 1\}$. We define the product $\boxcircle$ on $\mathbb{Q}A_z^{bi}$ by

$$\begin{aligned} z_{k_1,d_1} \boxcircle z_{k_2,d_2} :=& \binom{d_1+d_2}{d_1} \sum_{1 \leq j \leq k_1} \lambda^j_{k_1,k_2} z_{j,d_1+d_2} \\ &+ \binom{d_1+d_2}{d_1} \sum_{1 \leq j \leq k_2} \lambda^j_{k_2,k_1} z_{j,d_1+d_2} + \binom{d_1+d_2}{d_1} z_{k_1+k_2,d_1+d_2} \end{aligned}$$

and $\mathbb{Q}$-bilinear continuation to $\mathbb{Q}A_z^{bi}$. Here, the constant $\lambda^j_{a,b}$ is defined as

$$\lambda^j_{a,b} := (-1)^{b-1}\binom{a+b-j-1}{a-j}\frac{B_{a+b-j}}{(a+b-j)!}.$$

The map $\boxcircle$ is associative and commutative (see [4, Theorem 3.6 i)]), i.e., it induces a quasi-shuffle product $\boxast$. In the following theorem, part (i) is [4, Theorem 3.6 ii)] and part (ii) is from [4, Remark 3.8].

Theorem 10. (i) *The map* $\mathrm{g} : (\mathbb{Q}\langle A_z^{bi}\rangle, \boxast) \to (\mathcal{Z}_q, \cdot)$*, defined via*

$$z_{k_1,d_1} \cdots z_{k_r,d_r} \longmapsto \mathrm{g}\begin{pmatrix} k_1, \dots k_r \\ d_1, \dots, d_r \end{pmatrix},$$

$\mathbf{1} \mapsto 1$*, and $\mathbb{Q}$-linear continuation, is an algebra homomorphism.*

(ii) *The quasi-shuffle product $\boxast$ implies the stuffle product of MZVs.*

As for SZ qMZVs, also for bi-brackets, it is often convenient to work with their generating series. This was a central topic in Bachmann's Ph.D. thesis.

Theorem 11 ([4, Theorem 2.3]). *For $r \geq 1$, define*

$$\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} := \sum_{\substack{k_1,\dots,k_r>0\\ d_1,\dots,d_r>0}} \mathrm{g}\binom{k_1,\dots,k_r}{d_1-1,\dots,d_r-1} \prod_{j=1}^{r} X_j^{k_j-1} Y_j^{d_j-1}.$$

Then we have

$$\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \sum_{\substack{m_1>\dots>m_r>0\\ n_1,\dots,n_r\geq 1}} \prod_{j=1}^{r} e^{m_jY_j} e^{n_jX_j} q^{m_jn_j}.$$

On the level of generating series, one obtains a translation into the SZ model and vice versa.

Theorem 12 (Partition relation, [4, Theorem 2.3]). *For all $r \geq 1$ we have*

$$\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r} = \mathfrak{g}\binom{Y_1+\dots+Y_r,\dots,Y_1+Y_2,Y_1}{X_r,X_{r-1}-X_r,\dots,X_1-X_2}. \tag{2.2}$$

Remark 10. The name of this relation comes from the fact that $\mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$ is a sum over all partitions with exactly r distinct parts, and the relation itself is obtained by taking the sum over the partitions with conjugated Young diagram.

Another application of the generating series of bi-brackets is to give elegant translations between bi-brackets and the SZ model. That is possible since SZ qMZVs, as well as bi-brackets, span $\mathcal{Z}_q$ (Proposition 3, Theorem 9).

Theorem 13 (Translation bi-brackets-SZ model, [12, Theorem 2.18]).

(i) *For every $r \geq 1$ we have*

$$\prod_{j=1}^{r} e^{X_j} e^{Y_1+\dots+Y_j} \cdot \mathfrak{s}\binom{e^{X_1}-1,\dots,e^{X_r}-1}{e^{Y_1}-1,\dots,e^{Y_1+\dots+Y_r}-1} = \mathfrak{g}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}.$$

(ii) *For every $r \geq 1$ we have*

$$\mathfrak{s}\binom{X_1,\dots,X_r}{Y_1,\dots,Y_r}$$
$$= \left(\prod_{j=1}^{r}(1+X_j)(1+Y_j)\right)^{-1} \mathfrak{g}\binom{\ln(X_1+1),\dots,\ln(X_r+1)}{\ln(Y_1+1),\dots,\ln(Y_r+1)-\ln(Y_{r-1}+1)}.$$

Proof. (i) We are done by multiplying both sides in Theorem 4 with the term $\prod_{j=1}^{r}(1+X_j)(1+Y_j)$ and then substituting

$$X_j \longmapsto e^{X_j}-1, \quad Y_j \longmapsto e^{Y_1+\cdots+Y_j}-1 \qquad \text{for all } 1 \leq j \leq r.$$

(ii) We obtain the claim by substituting

$$X_j \longmapsto \ln(X_j+1), \quad Y_j \longmapsto \ln(Y_j+1)-\ln(Y_{j-1}+1) \quad (Y_0 := 0)$$

in (i), for all $1 \leq j \leq r$. □

Theorem 13 gives a deep connection between SZ qMZVs and bi-brackets.

Theorem 14 ([12, Theorem 3.22])**.** *SZ duality and the partition relation are equivalent.*

Remark 11. From Theorem 13, we also get a new proof of the well-known fact that bi-brackets and SZ qMZVs span the same $\mathbb{Q}$-vector space, $\mathcal{Z}_q$.

A direct but less elegant translation of bi-brackets into SZ qMZVs can be obtained using elementary calculations and identities like Equation (2.1).

Theorem 15 ([12, Theorem 2.19])**.** *For every* $r \in \mathbb{N}$*,* $k_1, \dots, k_r \in \mathbb{N}$*, and* $d_1, \dots, d_r \in \mathbb{N}_0$*, we have*

$$\begin{aligned}
&\mathrm{g}\binom{k_1,\dots,k_r}{d_1,\dots,d_r} \\
&= \sum_{\substack{1\leq n_j\leq p_j\leq k_j \\ 0\leq f_j\leq d_j \\ 1\leq j\leq r}} c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n}) \cdot \left[\prod_{j=1}^{r}\binom{d_j}{f_j}\binom{k_j-n_j}{p_j-n_j}\right] \\
&\qquad\qquad \times \prod_{j=1}^{r} \sum_{g_j=0}^{F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j)} \binom{F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j)}{g_j} \sum_{\ell_j=0}^{g_j} \left(\delta_{g_j=0} + \sum_{\substack{s_1+\cdots+s_{\ell_j}=g_j \\ s_i\geq 1}} \binom{g_j}{s_1,\dots,s_{\ell_j}}\right) \\
&\qquad\qquad \times \sum_{h_j=0}^{H_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j)} \binom{H_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j)}{h_j} \times \zeta_q^{\mathrm{SZ}}\left(p_1,\{0\}^{\ell_1},\dots,p_r,\{0\}^{\ell_r}\right),
\end{aligned}$$

with $c_{\mathbf{k}}^{\mathbf{d}}(\mathbf{n}) := \prod_{l=1}^{r} \frac{1}{d_\ell!(k_\ell-1)!}\left(\sum_{i=0}^{n_\ell-1}(-1)^i\binom{k_\ell}{i}(n_\ell-i)^{k_\ell-1}\right) \in \mathbb{Q}$ *and, for every* $1 \leq j \leq r$*,*

$$F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j) := f_j + \sum_{i=1}^{j-1}(f_i-g_i-h_i),$$

$$H_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j) := F_{\mathbf{h},\mathbf{g}}^{\mathbf{f}}(j) - g_j = f_j - g_j + \sum_{i=1}^{j-1}(f_i - g_i - h_i).$$

Example 1. We have, for example,

$$\begin{aligned}\mathrm{g}\binom{3,2}{0,2} = \frac{1}{4}\big(&\zeta_q^{\mathrm{SZ}}(1,1) + \zeta_q^{\mathrm{SZ}}(1,2) + 3\zeta_q^{\mathrm{SZ}}(2,1) + 3\zeta_q^{\mathrm{SZ}}(2,2) + 2\zeta_q^{\mathrm{SZ}}(3,1)\\ &+2\zeta_q^{\mathrm{SZ}}(3,2)\big) + \frac{3}{4}\big(\zeta_q^{\mathrm{SZ}}(1,1,0) + \zeta_q^{\mathrm{SZ}}(1,2,0)\\ &+3\zeta_q^{\mathrm{SZ}}(2,1,0) + 3\zeta_q^{\mathrm{SZ}}(2,2,0) + 2\zeta_q^{\mathrm{SZ}}(3,1,0) + 2\zeta_q^{\mathrm{SZ}}(3,2,0)\big)\\ &+\frac{1}{2}\big(\zeta_q^{\mathrm{SZ}}(1,1,0,0) + \zeta_q^{\mathrm{SZ}}(1,2,0,0) + 3\zeta_q^{\mathrm{SZ}}(2,1,0,0)\\ &+3\zeta_q^{\mathrm{SZ}}(2,2,0,0) + 2\zeta_q^{\mathrm{SZ}}(3,1,0,0) + 2\zeta_q^{\mathrm{SZ}}(3,2,0,0)\big).\end{aligned}$$

Remark 12. Zudilin's model of qMZVs, *multiple q-zeta brackets*, is closely related to bi-brackets (see [36] for details). They are defined for natural numbers $k_1, \ldots, k_r, d_1, \ldots, d_r \geq 1$ ($r \geq 0$) as

$$\mathfrak{z}_q\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} := c \sum_{\substack{m_1,\ldots,m_r>0\\ n_1,\ldots,n_r>0}} \prod_{j=1}^{r} m_j^{d_j-1} n_j^{k_j-1} q^{(m_1+\cdots+m_r)n_1+\cdots+m_r n_r},$$

with $c := \left(\prod_{j=1}^{r}(k_j-1)!(d_j-1)!\right)^{-1}$. Also, in this model, there is a duality relation,

$$\mathfrak{z}_q\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} = \mathfrak{z}_q\binom{d_r, \ldots, d_1}{k_r, \ldots, k_1},$$

which is exactly the partition relation for bi-brackets [36, Proposition 4].

We do not carry out a detailed study of this model since it is almost Bachmann's bi-bracket model, and the connection to bi-brackets is also reviewed very well in [36].

2.4. Takeyama-Bradley–Zhao Model

In Proposition 4, we saw that the $\mathbb{Q}$-span of BZ qMZVs is a proper subspace of $\mathcal{Z}_q$. However, it is comfortable for some situations to extend the BZ model of qMZVs, especially when the elements of the model should span $\mathcal{Z}_q$.

One such extension for the BZ model is the one due to Takeyama [30].

Definition 13. Set $\overline{\mathbb{N}} := \left\{\overline{1}\right\} \cup \mathbb{N} = \left\{\overline{1}, 1, 2, 3, \dots\right\}$ and define, for all $r \geq 1$, $k_1, \dots, k_r \in \overline{\mathbb{N}}$, $k_1 \neq 1$,

$$\zeta_q^{\mathrm{TBZ}}(k_1, \dots, k_r) := \sum_{m_1 > \cdots > m_r > 0} f(k_1, m_1) \cdots f(k_r, m_r),$$

where $f\left(\overline{1}, m\right) := \frac{q^m}{1-q^m},\ f(k, m) := \frac{q^{(k-1)m}}{(1-q^m)^k}$ for $k \geq 1$. Define $\zeta_q^{\mathrm{TBZ}}(\emptyset) := 1$.

As mentioned, this extension of the BZ model spans $\mathcal{Z}_q$.

Proposition 6. *The TBZ model spans $\mathcal{Z}_q$, i.e.,*

$$\mathcal{Z}_q = \left\langle \zeta_q^{\mathrm{TBZ}}(k_1, \dots, k_r) \,\middle|\, r \geq 0,\ k_i \in \overline{\mathbb{N}},\ k_1 \neq 1 \right\rangle_{\mathbb{Q}}.$$

Proof. The TBZ model spans the same space as the SZ model (Proposition 7, Proposition 9), i.e., $\mathcal{Z}_q$. □

This extended version of the BZ model satisfies a quasi-shuffle product that is compatible with the one of the non-extended model.

Definition 14. (i) Define $\mathfrak{h}^{\mathrm{TBZ}} := \mathbb{Q}\langle z_{\overline{1}}, z_1, z_2, \dots\rangle$. Then we can view ζ_q^{TBZ} also as the map

$$\zeta_q^{\mathrm{TBZ}} : \mathfrak{h}^{\mathrm{TBZ}} \longrightarrow \mathcal{Z}_q, \qquad z_{k_1} \cdots z_{k_r} \longmapsto \zeta_q^{\mathrm{TBZ}}(k_1, \dots, k_r),$$

extended $\mathbb{Q}$-linearly, and sending $\mathbf{1} \mapsto 1$.
(ii) Define on the alphabet $\mathbb{Q}\left\{z_k \,\middle|\, k \in \overline{\mathbb{N}}\right\}$ the associative and commutative product $\diamond_{\mathrm{TBZ}}$ via

$$\begin{aligned} z_{k_1} \diamond_{\mathrm{TBZ}} z_{k_2} :=& z_{k_1+k_2} + z_{k_1+k_2-1}, \\ z_k \diamond_{\mathrm{TBZ}} z_{\overline{1}} := z_{\overline{1}} \diamond_{\mathrm{TBZ}} z_k :=& z_{k+1}, \quad z_{\overline{1}} \diamond_{\mathrm{TBZ}} z_{\overline{1}} := z_2 - z_1 \end{aligned}$$

for all $k, k_1, k_2 \in \mathbb{N}$. Furthermore, let $*_{\mathrm{TBZ}}$ be the induced quasi-shuffle product on $\mathbb{Q}\langle z_{\overline{1}}, z_1, z_2, \dots\rangle$.

Some straightforward computation shows that $\diamond_{\mathrm{TBZ}}$ is commutative and associative.

Proposition 7. *The map ζ_q^{TBZ} is an algebra homomorphism, i.e., for all $u, v \in \mathbb{Q}\langle z_{\overline{1}}, z_1, z_2, \dots\rangle$ we have*

$$\zeta_q^{\mathrm{TBZ}}(u *_{\mathrm{TBZ}} v) = \zeta_q^{\mathrm{TBZ}}(u)\zeta_q^{\mathrm{TBZ}}(v).$$

Proof. The proof is analogous to the proof of Proposition 5. □

Remark 13. For the TBZ model, no “good” generating series is known. Hence, it remains to determine one that can be written nicely or would give us new results about TBZ qMZVs.

Proposition 8 ([30, Theorem 4])**.** *Let the maps* $\mathrm{U}_{\mathrm{TBZ}}$ *and* $\mathrm{V}_{\mathrm{TBZ}}$ *be as in Proposition 9. Then, on* $\mathfrak{h}^{\mathrm{TBZ}}$ *we have*

$$\zeta_q^{\mathrm{TBZ}} = \zeta_q^{\mathrm{TBZ}} \circ \mathrm{V}_{\mathrm{TBZ}} \circ \tilde{\tau} \circ \mathrm{U}_{\mathrm{TBZ}} \,.$$

Proof. Since $\mathrm{U}_{\mathrm{TBZ}}$ and $\mathrm{V}_{\mathrm{TBZ}}$ are translation maps of TBZ qMZVs into the SZ model, respectively vice versa, the proof follows by SZ duality. □

We now give a direct translation into the SZ model and vice versa.

Proposition 9 ([12, Proposition 2.23])**.** (i) *For all tuples of integers* $d_1, \ldots, d_r \in \mathbb{N}_0$*, and* $k_1, \ldots, k_{r-1} \in \mathbb{N}$ *with* $k_1 \geq 2$ *if* $d_1 = 0$*, we have*

$$\begin{aligned}
&\zeta_q^{\mathrm{TBZ}}\left(\{\overline{1}\}^{d_1}, k_1, \ldots, k_{r-1}, \{\overline{1}\}^{d_r}\right) \\
&\qquad = \sum_{\substack{\delta_j \in \{0,1\} \\ 1 \leq j \leq r-1}} \zeta_q^{\mathrm{SZ}}\left(\{1\}^{d_1}, k_1 - \delta_1, \ldots, \{1\}^{d_{r-1}}, k_{r-1} - \delta_{r-1}, \{1\}^{d_r}\right).
\end{aligned}$$

Denote the corresponding $\mathbb{Q}$*-linear map* $\mathfrak{h}^{\mathrm{TBZ}} \to \mathfrak{R}^1$ *with* $\mathbf{1} \mapsto 1$ *by* $\mathrm{U}_{\mathrm{TBZ}}$*.*
(ii) *For every SZ admissible index* $\mathbf{k} = \left(k_1, \{0\}^{d_1}, \ldots, k_r, \{0\}^{d_r}\right)$*, we have*

$$\begin{aligned}
\zeta_q^{\mathrm{SZ}}(\mathbf{k}) = &\sum_{\substack{1 \leq j_i \leq k_i, \\ \varepsilon_i \in \{\overline{1}, 1\}^{d_i} \\ 1 \leq i \leq r}} (-1)^{\sum_{i=1}^{r} k_i - j_i + |\varepsilon_i|} \\
&\qquad \times \zeta_q^{\mathrm{TBZ}}\left(j_1 \delta_{j_1 \neq 1} + \overline{1}\delta_{j_1 = 1}, \varepsilon_1, \ldots, j_r \delta_{j_r \neq 1} + \overline{1}\delta_{j_r = 1}, \varepsilon_r\right),
\end{aligned}$$

where $|\varepsilon|$ *counts the* $\overline{1}$*’s in* ε*; we denote the corresponding map* $\mathfrak{R}^1 \to \mathfrak{h}^{\mathrm{TBZ}}$ *with* $\mathbf{1} \mapsto 1$ *by* $\mathrm{V}_{\mathrm{TBZ}}$*.*

Proof. (i) is a consequence of the identity $\frac{X^{k-1}}{(1-X)^k} = \frac{X^{k-1}}{(1-X)^{k-1}} + \frac{X^k}{(1-X)^k}$ for $k \in \mathbb{N}$, while *(ii)* follows from the identities $1 = \frac{1}{1-X} - \frac{X}{1-X}$ and

$$\frac{X^k}{(1-X)^k} = \sum_{1 \leq j \leq k} (-1)^{k-j} \left[\frac{X^{j-1}}{(1-X)^j}\delta_{j \neq 1} + \frac{X}{1-X}\delta_{j=1}\right] \quad \text{for } k \in \mathbb{N}.$$ □

2.5. Ohno–Okuda–Zudilin Model

Another model of q-analogs of MZVs is the one first considered in 2012 by Ohno, Okuda, and Zudilin [22]. One application of this model is that a particular sum of OOZ qMZVs is the generating series of the number of conjugacy classes of $\mathrm{GL}(n, K)$ for a finite field K (cf. [12, Section 4.6]).

Definition 15. (i) Define $\zeta_q^{\mathrm{OOZ}}(\emptyset) := 1$, and, for $r \geq 1$, and every SZ admissible index $(k_1, \ldots, k_r)$,

$$\begin{aligned}\zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_r) :=& \zeta_q(k_1, \ldots, k_r; X, 1 \ldots, 1)\\ =& \sum_{m_1 > \cdots > m_k > 0} q^{m_1} \prod_{j=1}^{r} \frac{1}{(1-q^{m_j})^{k_j}}.\end{aligned}$$

(ii) On an algebraic level, ζ_q^{OOZ} can be seen as an evaluation map $\zeta_q^{\mathrm{OSZ}} : \mathfrak{K}^1 \to \mathcal{Z}_q$, defined via $\mathbf{1} \mapsto 1$, $\mathbb{Q}$-linearity, and

$$p^{k_1} y \cdots p^{k_r} y \longmapsto \zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_r).$$

(iii) By restricting to admissible indices, define $\zeta_q^{\mathrm{OBZ}} : \mathfrak{h}^0 \to \mathcal{Z}_q$, given through $\mathbb{Q}$-linearity, $\mathbf{1} \mapsto 1$, and

$$z_{k_1} \cdots z_{k_r} \longmapsto \zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_r).$$

Considering the span of both OOZ models, the link to SZ qMZVs (respectively BZ qMZVs) becomes clear.

Proposition 10. *For the $\mathbb{Q}$-span of the OOZ model we have*

(i) $\mathcal{Z}_{q,1} = \left\langle \zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_r) \,|\, r \geq 0,\ k_1 \geq 2,\ k_i \geq 1 \right\rangle_{\mathbb{Q}}$,

(ii) $\mathcal{Z}_q = \left\langle \zeta_q^{\mathrm{OOZ}}(k_1, \ldots, k_r) \,|\, r \geq 0,\ k_1 \geq 1,\ k_i \geq 0 \right\rangle_{\mathbb{Q}}$.

Proof. A proof can be obtained from Proposition 12, where explicit translations of OOZ qMZVs into SZ qMZVs and vice versa (respectively restricted OOZ qMZVs into BZ qMZVs) are given. □

In particular, the OOZ model we work with is closed under $q\frac{d}{dq}$ (Remark 3(i)), while the restricted one is only conjecturally closed (Remark 3(ii)). The restricted model satisfies a shuffle product (as shown in Proposition 4.5 in [14]).

Definition 16. Define the map $T : \mathfrak{h}^0 \to \mathfrak{h}^1$ via $\mathbf{1} \mapsto 1$, $\mathbb{Q}$-linearity, and

$$z_n v \longmapsto z_n v - z_{n-1} v$$

for all $n \geq 2$, $v \in \mathfrak{h}^1$. Then the quasi-shuffle product $\ast_{\mathrm{OOZ}}$ on $\mathfrak{h}^0$ for the OOZ model is defined as the unique map $\mathfrak{h}^0 \times_{\mathbb{Q}} \mathfrak{h}^0 \to \mathfrak{h}^0$ satisfying

$$T(u \ast_{\mathrm{OOZ}} v) = T(u) * T(v)$$

for all $u, v \in \mathfrak{h}^0$.

The quasi-shuffle product $\ast_{\mathrm{OOZ}}$ turns ζ_q^{OBZ} into a homomorphism on $\mathfrak{h}^0$.

Theorem 16 ([14, Proposition 4.5])**.** *The product* $\ast_{\mathrm{OOZ}}$ *is well-defined, and the evaluation map*

$$\zeta_q^{\mathrm{OBZ}} : (\mathfrak{h}^0, \ast_{\mathrm{OOZ}}) \longrightarrow (\mathcal{Z}_q, \cdot), \qquad w \longmapsto \zeta_q^{\mathrm{OBZ}}(w)$$

is an algebra homomorphism; in particular, for all $u, v \in \mathfrak{h}^0$, *we have*

$$\zeta_q^{\mathrm{OBZ}}(u \ast_{\mathrm{OOZ}} v) = \zeta_q^{\mathrm{OBZ}}(u)\zeta_q^{\mathrm{OBZ}}(v).$$

For details on the quasi-shuffle structure that OOZ qMZVs imply, we refer to [14] and [15]. As for other models, we consider a generating series for (the extended version of) OOZ qMZVs.

Proposition 11 ([12, Proposition A.89])**.** *Define for all* $r \in \mathbb{N}_0$

$$\mathfrak{oz}\binom{\mathbf{X}_r}{\mathbf{Y}_r} := \sum_{\substack{k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 0}} \zeta_q^{\mathrm{OOZ}}\left(k_1, \{0\}^{d_1}, \dots, k_r, \{0\}^{d_r}\right) \frac{X_1^{k_1}}{k_1!} Y_1^{d_1} \cdots \frac{X_r^{k_r}}{k_r!} Y_r^{d_r}.$$

Then, with $m_{r+1} := 0$ *we have*

$$\mathfrak{oz}\binom{\mathbf{X}_r}{\mathbf{Y}_r} = \sum_{m_1 > \cdots > m_r > 0} q^{m_1} \prod_{j=1}^{r} (1 + Y_j)^{m_j - m_{j+1} - 1} \left(e^{\frac{X_j}{1-q^{m_j}}} - 1\right).$$

Proof. We use the geometric sum identity and the binomial theorem to obtain

$$\begin{aligned}
\mathfrak{oz}\binom{\mathbf{X}_r}{\mathbf{Y}_r} &= \sum_{\substack{k_1,\dots,k_r \geq 1 \\ d_1,\dots,d_r \geq 0}} \sum_{\substack{m_1 > n_1 > \cdots > n_{d_1} \\ > m_2 > \cdots > n_{d_1+\cdots+d_r} > 0}} q^{m_1} \prod_{j=1}^{r} \frac{1}{(1-q^{m_j})^{k_j}} \frac{X_j^{k_j}}{k_j!} Y_j^{d_j} \\
&= \sum_{m_1 > \cdots > m_r > 0} q^{m_1} \prod_{j=1}^{r} \left(\sum_{k_j \geq 1,\, d_j \geq 0} \frac{\left(\frac{X_j}{1+q^{m_j}}\right)^{k_j}}{k_j!} \binom{m_j - m_{j+1} - 1}{d_j} Y_j^{d_j} \right) \\
&= \sum_{m_1 > \cdots > m_r > 0} q^{m_1} \prod_{j=1}^{r} (1 + Y_j)^{m_j - m_{j+1} - 1} \left(e^{\frac{X_j}{1-q^{m_j}}} - 1\right). \qquad \square
\end{aligned}$$

For the OOZ model, no particular duality relation is known so far. However, we can translate into the SZ model (respectively BZ model for restricted definition), then apply SZ duality (respectively BZ duality), and translate back into the OOZ model, giving $\mathbb{Q}$-linear relations among OOZ qMZVs.

Theorem 17 ([15, Theorem 5.9])**.** *Let* U, V, *be as in Proposition 12.*

(i) *On* $\mathfrak{h}^0$ *we have* $\zeta_q^{\mathrm{OBZ}} = \zeta_q^{\mathrm{OBZ}} \circ \mathrm{U}^{-1} \circ \tau \circ \mathrm{U}$.

(ii) *On* $\mathfrak{K}^1$ *we have* $\zeta_q^{\mathrm{OSZ}} = \zeta_q^{\mathrm{OSZ}} \circ \mathrm{V}^{-1} \circ \tilde{\tau} \circ \mathrm{V}$.

(iii) *On* $\mathfrak{K}^1$ *we have* $\zeta_q^{\mathrm{OSZ}} = \zeta_q^{\mathrm{SZ},\star} \circ \tilde{\tau}$, *where* $\zeta_q^{\mathrm{SZ},\star} : \mathfrak{K}^1 \to \mathcal{Z}_q$ *is the map of SZ-star qMZVs, defined via* $\mathbf{1} \mapsto 1$, *$\mathbb{Q}$-linearity and*

$$p^{k_1}y \cdots p^{k_r}y \longmapsto \sum_{m_1 \geq \cdots \geq m_r > 0} \frac{q^{m_1 k_1}}{(1-q^{m_1})^{k_1}} \cdots \frac{q^{m_r k_r}}{(1-q^{m_r})^{k_r}}.$$

As mentioned, we now give the maps U and V, proving in particular Proposition 10.

Proposition 12 ([15, Proposition 5.7])**.** (i) *The $\mathbb{Q}$-linear map* $\mathrm{U} : \mathfrak{h}^0 \to \mathfrak{h}^0$, *satisfying* $\mathbf{1} \mapsto 1$ *and*

$$z_{k_1} \cdots z_{k_r} \longmapsto \sum_{\substack{2 \leq n_1 \leq k_1 \\ 1 \leq n_j \leq k_j,\, j \geq 2}} \binom{k_1-2}{n_1-2}\binom{k_2-1}{n_2-1} \cdots \binom{k_r-1}{n_r-1} z_{n_1} \ldots z_{n_r},$$

is a linear isomorphism with $\zeta_q^{\mathrm{OBZ}} = \zeta_q^{\mathrm{BZ}} \circ \mathrm{U}$.

(ii) *Analogously, the map* $\mathrm{V} : \mathfrak{K}^1 \to \mathfrak{K}^1$, *given by* $\mathbf{1} \mapsto 1$,

$$p^{k_1}y \cdots p^{k_r}y \longmapsto \sum_{\substack{1 \leq n_1 \leq k_1 \\ 0 \leq n_j \leq k_j,\, j \geq 2}} \binom{k_1-1}{n_1-1}\binom{k_2}{n_2} \cdots \binom{k_r}{n_r} p^{n_1}y \ldots p^{n_r}y,$$

and $\mathbb{Q}$-linear continuation is a linear isomorphism with $\zeta_q^{\mathrm{OSZ}} = \zeta_q^{\mathrm{SZ}} \circ \mathrm{V}$.

Remark 14. (i) The "duality relations" in Proposition 12 (i) and (ii) can indeed be viewed as duality in the OOZ model. This was done by Ebrahimi-Fard, Manchon, and Singer [15]. Still, we should compare them with the partition relation in the bi-bracket model. The partition relation is the same when translating bi-brackets into SZ qMZVs, applying SZ duality, and then translating back into bi-brackets.

(ii) Duality relation (iii) is no "real" duality relation, but in Theorem 5.5 in [15] it is called so, since $\tilde{\tau}$ gives SZ duality in the SZ model. However, this relation is the translation map of the OOZ model into the SZS model.

2.6. Okounkov qMZVs

In the context of enumerative geometry and Hilbert schemes, a model of qMZVs introduced by Okounkov [23] occurs often.

Definition 17 (Okounkov qMZVs)**.** For all $\mathbf{k} = (k_1, \dots, k_r) \in \mathbb{N}^r_{\geq 2}$ and some $r \in \mathbb{N}_0$, its *Okounkov qMZV* is

$$\begin{aligned} \zeta_q^{\mathrm{Oko}}(\mathbf{k}) := \zeta_q^{\mathrm{Oko}}(k_1, \dots, k_r) &:= \zeta_q\left(k_1, \dots, k_r; p_{k_1}, \dots, p_{k_r}\right) \\ &= \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{p_{k_j}(q^{m_j})}{(1-q^{m_j})^{k_j}}, \end{aligned}$$

where $\zeta_q^{\mathrm{Oko}}(\emptyset) := 1$ as usual and

$$p_k(X) := \begin{cases} X^{\frac{k}{2}} & \text{if } k \text{ is even,} \\ X^{\frac{k-1}{2}}(1+X) & \text{if } k \text{ is odd.} \end{cases}$$

The span of the Okounkov model is a proper subspace of $\mathcal{Z}_q$ as noted in (iv) of page 7 in [8].

Proposition 13 ([8])**.** *We have*

$$\mathcal{Z}^{\circ}_{q,1} = \left\langle \zeta_q^{\mathrm{Oko}}(k_1, \dots, k_r) \,|\, r \geq 0,\, k_i \geq 2 \right\rangle_{\mathbb{Q}}.$$

In particular, the span of Okounkov qMZVs is conjecturally closed under $q\frac{d}{dq}$ (Remark 3 (ii)).

As the other models of qMZVs, Okounkov qMZVs also satisfy a quasi-shuffle product.

Definition 18. (i) Define $\mathfrak{h}^{\mathrm{Oko}} := \mathbb{Q}\langle z_2, z_3 \dots \rangle$ and $\zeta_q^{\mathrm{Oko}} : \mathfrak{h}^{\mathrm{Oko}} \to \mathcal{Z}_q$ via $\mathbb{Q}$-linearity, $\mathbf{1} \mapsto 1$ and

$$z_{k_1} \cdots z_{k_r} \longmapsto \zeta_q^{\mathrm{Oko}}(k_1, \dots, k_r).$$

(ii) Define on $A_{\mathrm{Oko}} := \mathbb{Q}\{z_2, z_3, \dots\}$ the product $\diamond_{\mathrm{Oko}}$ by $\mathbb{Q}$-bilinearity, $\mathbf{1} \diamond_{\mathrm{Oko}} w := w \diamond \mathbf{1}$ for all $w \in A_{\mathrm{Oko}}$ and

$$(z_{k_1}, z_{k_2}) \longmapsto \begin{cases} z_{k_1+k_2-1} + z_{k_1+k_2+1}, & \text{if } k_1,\, k_2 \text{ is odd,} \\ z_{k_1+k_2}, & \text{otherwise.} \end{cases}$$

Furthermore, let $*_{\mathrm{Oko}}$ be the induced quasi-shuffle product on $\mathfrak{h}^{\mathrm{Oko}}$.

Proposition 14. *The map ζ_q^{Oko} is an algebra homomorphism; In particular, for all $w_1, w_2 \in \mathfrak{h}^{\mathrm{Oko}}$ we have*

$$\zeta_q^{\mathrm{Oko}}(w_1 *_{\mathrm{Oko}} w_2) = \zeta_q^{\mathrm{Oko}}(w_1)\zeta_q^{\mathrm{Oko}}(w_2).$$

Proof. The claim follows by the definition of Okounkov qMZVs as iterated sums. If at least one of k_1 and k_2 is even, then we already have $p_{k_1}(X)p_{k_2}(X) = p_{k_1+k_2}(X)$ and if both are odd, we have

$$\begin{aligned} p_{k_1}(X)p_{k_2}(X) &= \left(X^{\frac{(k_1+k_2-1)-1}{2}} + X^{\frac{(k_1+k_2+1)-1}{2}}\right)(1+X) \\ &= p_{k_1+k_2-1}(X) + p_{k_1+k_2+1}(X). \end{aligned}$$ □

Remark 15. For the Okounkov model, there is no "good" generating series known, i.e., one that has a nice representation or would lead to further results. However, we can introduce one for $r \geq 1$:

$$\begin{aligned} \mathfrak{o}(X_1, \ldots, X_r) &:= \sum_{k_1,\ldots,k_r \geq 2} \zeta_q^{\mathrm{Oko}}(k_1, \ldots, k_r) X_1^{k_1-2} \cdots X_r^{k_r-2} \\ &= \sum_{m_1 > \cdots > m_r > 0} \prod_{j=1}^{r} \frac{q^{m_j}(1 - q^{m_j} + (1 + q^{m_j})X_j)}{(1 - q^{m_j})((1 - q^{m_j})^2 - q^{m_j} X_j)}. \end{aligned}$$

Remark 16. Since the Okounkov model does not span the same space as the SZ or BZ model, we cannot translate into the respective model, apply SZ or BZ duality, and translate back as we did in the OOZ model. Also, no particular duality relation for the Okounkov model is known.

3. Subalgebras of $\mathcal{Z}_q$

Several subalgebras of $\mathcal{Z}_q$ are interesting. One of the most important is the algebra of quasi-modular forms. Others take their importance from conjectures stating that they are not only subalgebras but also equal $\mathcal{Z}_q$, which would give — assuming that they are true — a much deeper understanding of the structure of $\mathcal{Z}_q$. They are all verified for small weights, often with computer assistance. For example, before he introduced bi-brackets, Bachmann considered brackets and their algebra

$$\mathcal{MD} := \langle \mathrm{g}(k_1, \ldots, k_r) \mid r \geq 0,\, k_1 \geq 2,\, k_i \geq 1 \rangle_{\mathbb{Q}}\,.$$

The algebra $\mathcal{MD}$ contains the classical Eisenstein series G_2, G_4, G_6 because of

$$G_2 = -\frac{1}{24} + \mathrm{g}(2), \quad G_4 = \frac{1}{1440} + \mathrm{g}(4), \quad G_6 = -\frac{1}{60480} + \mathrm{g}(6).$$

Hence, by Proposition 1 in [20], the ring of quasi-modular forms $\widetilde{M}(\mathrm{SL}_2(\mathbb{Z}))_{\mathbb{Q}}$ is contained in $\mathcal{MD}$;

$$\widetilde{M}(\mathrm{SL}_2(\mathbb{Z}))_{\mathbb{Q}} = \mathbb{Q}[G_2, G_4, G_6] \subset \mathcal{MD}.$$

Notice that this is a proper inclusion since, e.g., $\mathrm{g}(2,1) \neq 0$ has odd weight and hence, $\mathrm{g}(2,1)$ is not algebraic over $\mathbb{Q}$ in terms of G_2, G_4, G_6. We already mentioned $\mathcal{MD}$ with a different name.

Proposition 15 ([8, Theorem 1 (ii)]). *We have* $\mathcal{MD} = \mathcal{Z}_q^{\circ}$. *In particular,* $\mathcal{MD}$ *is stable under* $q\frac{d}{dq}$.

For the $\mathbb{Q}$-algebra of bi-brackets (often denoted by $\mathcal{BD}$) it is proven in [8, Theorem 1 (i)] that $\mathcal{BD} = \mathcal{Z}_q$. By comparing dimensions in small weights, Bachmann conjectured that brackets and bi-brackets span the same space.

Conjecture 1 ([4, Conjecture 4.3]). We have $\mathcal{MD} = \mathcal{BD}$, i.e., $\mathcal{Z}_q^{\circ} = \mathcal{Z}_q$.

The following is another statement about some subalgebras of $\mathcal{Z}_q$.

Proposition 16. *We have*

$$\mathcal{Z}_{q,1}^{\circ} = \langle \mathrm{g}(\mathbf{k}) \,|\, k_i \geq 2 \rangle_{\mathbb{Q}} = \left\langle \zeta_q^{\mathrm{Oko}}(\mathbf{k}) \,|\, k_i \geq 2 \right\rangle_{\mathbb{Q}} = \left\langle \zeta_q^{\mathrm{BZ}}(\mathbf{k}) \,|\, k_i \geq 2 \right\rangle_{\mathbb{Q}}.$$

Proof. This is proven using [8, Theorem 2.3 (iii)], Proposition 13, and an analogous proof of Proposition 4. □

We get other important subalgebras of $\mathcal{Z}_q$ when considering bi-brackets. By defining the weight and depth as in Definition 11, we get a filtration by weight and depth on $\mathcal{Z}_q$ (respectively on every subalgebra of $\mathcal{Z}_q$).

Definition 19 ([4, Definition 4.1]). Let be A a subalgebra of $\mathcal{Z}_q$ and $r, s \geq 0$. Define

(i) the *weight filtration*

$$\mathrm{Fil}_r^W(A) := \left\langle b = \mathrm{g}\begin{pmatrix} k_1, \dots, k_s \\ d_1, \dots, d_s \end{pmatrix} \in A \,\middle|\, 0 \leq s \leq r,\ \mathrm{wt}(b) \leq r \right\rangle_{\mathbb{Q}},$$

(ii) the *depth filtration* $\mathrm{Fil}_k^D(A) := \left\langle b = \mathrm{g}\binom{k_1,\dots,k_s}{d_1,\dots,d_s} \in A \,\middle|\, 0 \leq s \leq r \right\rangle_{\mathbb{Q}}$,

(iii) $\mathrm{Fil}_{r,s}^{W,D}(A) := \mathrm{Fil}_r^W \mathrm{Fil}_s^D(A)$,

and denote by gr_r^W, respectively $\mathrm{gr}_{r,s}^{W,D}$ the associated graded $\mathbb{Q}$-vector spaces.

For the dimensions of the graded parts of $\mathcal{Z}_q$, Bachmann and Kühn give in [8] conjectures standing in analogy to the one by Zagier and Broadhurst–Kreimer. Hence, for completeness, we will state the latter ones first. For all $k \geq 2$, $n \geq 2$, and $d \geq 0$, we use the notation

$$\begin{aligned}\mathcal{Z} &:= \langle \zeta(\mathbf{k}) | \, \mathbf{k} \text{ admissible} \rangle_{\mathbb{Q}}, \quad \mathcal{Z}_k := \langle \zeta(\mathbf{k}) | \, \operatorname{wt}(\mathbf{k}) = k \rangle_{\mathbb{Q}}, \\ \mathcal{Z}_n^d &:= \langle \zeta(\mathbf{k}) | \, \operatorname{wt}(\mathbf{k}) = n, \operatorname{depth}(\mathbf{k}) = d \rangle_{\mathbb{Q}}.\end{aligned}$$

Conjecture 2. (i) (Zagier). We have $\mathcal{Z} = \bigoplus_{k \geq 0} \mathcal{Z}_k$. Define d_k via

$$\sum_{k \geq 0} d_k X^k = \frac{1}{1 - X^2 - X^3} = \frac{1}{1 - x^2} \frac{1}{1 - O_3(x)}$$

with $O_3(X) := \frac{X^3}{1-X^2}$. Then we have $\dim_{\mathbb{Q}}(\mathcal{Z}_k) = d_k$.
(ii) (Hoffman). The MZVs of indices containing only 2's and 3's build a basis of $\mathcal{Z}$.

Note that Hoffman's conjecture is stronger than Zagier's and is in accordance with Brown's theorem stating that MZVs with indices containing only 2's and 3's generate $\mathcal{Z}$, i.e., $\dim_{\mathbb{Q}}(\mathcal{Z}_k) \leq d_k$.

Conjecture 3 ([13, Equation (7)])**.** With $E_2(X) := \frac{X^2}{1-X^2}$ and $S(X) := \frac{X^{12}}{(1-X^4)(1-X^6)}$, we have

$$1 + \sum_{n \geq 1, \, d \geq 1} \dim_{\mathbb{Q}} \left(\mathcal{Z}_n^d \big/ \mathcal{Z}_n^{d-1} \right) X^n Y^d = \frac{1 + E_2(X) Y}{1 - O_3(X) Y + S(X) Y^2 - S(X) Y^4}.$$

Conjecture 4 ([8, Conjecture 3])**.** (i) The dimensions of the weight graded parts of $\mathcal{Z}_q$ are given through

$$\begin{aligned}\sum_{k \geq 0} \dim_{\mathbb{Q}} \left(\operatorname{gr}_k^W \mathcal{Z}_q \right) X^k &= \frac{1}{1 - X - X^2 - X^3 + X^6 + X^7 + X^8 + X^9} \\ &= \frac{1}{(1-X^2)(1-X^4)(1-X^6)} \\ &\quad \times \frac{1}{1 - D(X) O_1(X) + D(X)(E_4(X) + 2S(X))},\end{aligned}$$

where we set $D(X) := \frac{1}{1-X^2}$, $O_1(X) := \frac{X}{1-X^2}$, $E_4(X) := \frac{X^4}{1-X^2}$.
(ii) Bi-brackets with indices only containing 1's, 2's, and 3's generate $\mathcal{Z}_q$.

(iii) For the weight and depth graded parts of $\mathcal{Z}_q$, we have

$$\sum_{k,\,\ell\geq 0} \dim_{\mathbb{Q}}\left(\mathrm{gr}^{W,D}_{k,\ell}\,\mathcal{Z}_q\right) X^k Y^\ell = \frac{1+D(X)E_2(X)Y+D(X)S(X)Y^2}{1-a_1(X)Y+a_2(X)Y^2-a_3(X)Y^3-a_4(X)Y^4+a_5(X)Y^5}$$

with

$$a_1(X):=D(X)O_1(X), \quad a_2(X):=D(X)\sum_{k\geq 1}\dim_{\mathbb{Q}}(M_k(\mathrm{SL}_2(\mathbb{Z})))^2X^k,$$
$$a_3(X)=a_5(X):=O_1(X)S(X),\ a_4(X):=D(X)\sum_{k\geq 1}\dim_{\mathbb{Q}}(S_k(\mathrm{SL}_2(\mathbb{Z})))^2X^k.$$

Okounkov conjectured the following about the structure of the qMZVs named after him.

Conjecture 5 ([23, Conjecture 1])**.** (i) We have

$$\sum_{k\geq 0}\dim_{\mathbb{Q}}\left(\mathrm{gr}^w_k\left(\mathcal{Z}_q^{Oko}\right)\right)t^k = \frac{1}{1-t^2-t^3-t^4-t^5+t^8+t^9+t^{10}+t^{11}+t^{12}}$$
$$= \frac{1}{(1-t^2)(1-t^4)(1-t^6)} \times \frac{1}{1-D(t)O_3(t)+2D(t)S(t)}.$$

(ii) The space $\mathcal{Z}_q^{\mathrm{Oko}}$ is spanned by $\zeta_q^{\mathrm{Oko}}(\mathbf{k})$ with $2\leq k_i\leq 5$.

Finally, with Figure 3, we give an overview of the diverse, most commonly considered, subalgebras of $\mathcal{Z}_q$. Note the following remark on Figure 3.

Remark 17. (i) We denote by **MF** the algebra of modular forms, where we take a modular form formally via its Fourier expansion in a canonical way as an element of the larger algebras.

(ii) The notation $q\frac{d}{dq}$ indicates that the respective algebra is closed under $q\frac{d}{dq}$. Dashed arcs with question marks mean it is not proven yet but conjectured.

(iii) An equality sign with a question mark means that equality is conjectured but not proven yet.

(iv) Equalities in blue boxes are different notations from several papers for the same algebra.

(v) Indices **k** can have arbitrary non-negative length. For the sake of clarity, we do not state this explicitly.

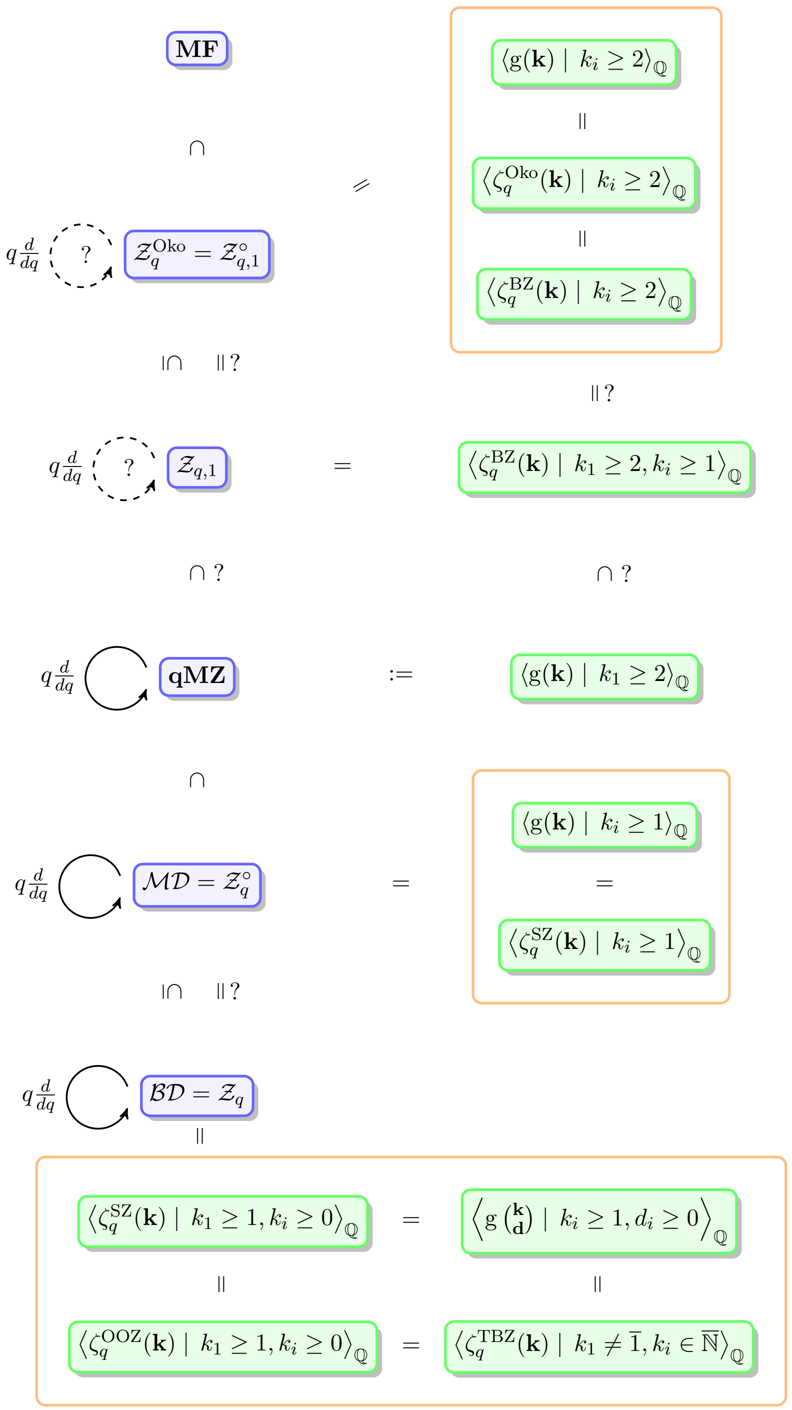


Figure 1: Subalgebras of $\mathcal{Z}_q$

4. qMZVs as Generating Functions of Marked Partitions

We give in this section a combinatorial view of the considered dualities of q-analogs of MZVs. For that, we will use partitions intensively. A good reference on partitions, in general, is [16]. For Stanley coordinates, we refer to Stanley's original work [29].

A partition of a natural number N is usually defined as a decreasing tuple of natural numbers $\lambda = (\lambda_1, \dots, \lambda_h)$ (i.e., $\lambda_1 \geq \cdots \geq \lambda_h$) with

$$|\lambda| := \lambda_1 + \cdots + \lambda_h = N.$$

We often write $\lambda \vdash N$, meaning that λ is a partition of N.

We can also characterize $\lambda \vdash N$ in a different way via summarizing the λ_i that are equal. Namely, we can identify λ with two tuples of natural numbers, $\mathbf{m} = (m_1, \dots, m_r)$ and $\mathbf{n} = (n_1, \dots, n_r)$, where $\mathbf{m}$ contains the values of λ, without repetitions, in strict descending order (i.e., $m_1 > \cdots > m_r > 0$) and $\mathbf{n}$ their multiplicities, i.e., n_i describes the number of λ_j being equal to m_i and one has $N = \sum_{j=1}^{r} m_j n_j$.

Definition 20 (Stanley coordinates)**.** A partition $\mathbf{p}$ of length r of some $N \in \mathbb{N}$ in *Stanley coordinates* is a pair of two r-tuples of natural numbers $(\mathbf{m}, \mathbf{n}) = ((m_1, \dots, m_r), (n_1, \dots, n_r))$ such that

(i) $m_1 > \cdots > m_r > 0$,

(ii) $m_1 n_1 + \cdots + m_r n_r = N$.

By $\mathbf{p}'$, we denote the *conjugated partition* of $\mathbf{p}$, i.e., the one with the Young diagram reflected at the main diagonal. Formally, if $\mathbf{p} = (\mathbf{m}, \mathbf{n}) \vdash N$ is a partition of a natural number N, we set

$$\mathbf{p}' := ((n_1 + \cdots + n_r, \dots, n_1 + n_2, n_1), (m_r, m_r - m_{r-1} \dots, m_1 - m_2)) \vdash N.$$

Figure 2 illustrates the formal definition of the conjugated partition (figure from [6]).

We will often consider sums over all partitions of a fixed number and with a fixed length. For this, we give the following definitions.

Definition 21. Define for every $N \in \mathbb{N}$ and $r \geq 1$ the set $\mathcal{P}_r(N)$ of partitions of N of length r, as

$$\left\{ ((m_1, \dots, m_r), (n_1, \dots, n_r)) \in \mathbb{N}^r \times \mathbb{N}^r \,\middle|\, m_1 > \cdots > m_r, \sum_{j=1}^{r} m_j n_j = N \right\},$$

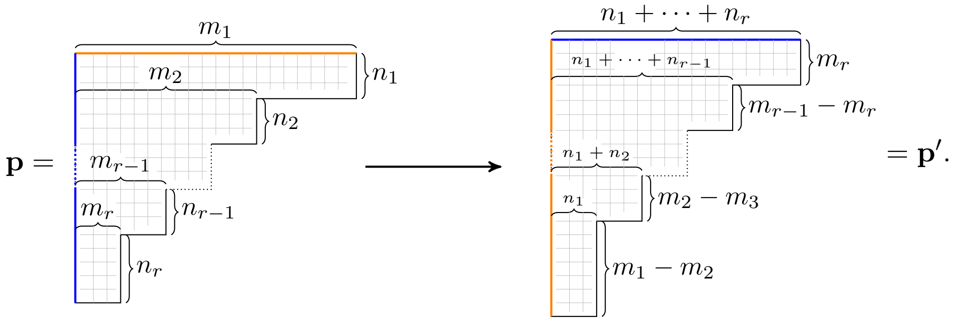


Figure 2: Conjugating a partition

and with analogous notation,

$$\mathcal{P}_{\le r}(N) := \bigcup_{s=1}^{r} \mathcal{P}_s(N),\ \mathcal{P}_r := \bigcup_{N>0} \mathcal{P}_r(N),\ \mathcal{P}_{\le r} := \bigcup_{N>0} \mathcal{P}_{\le r}(N),\ \mathcal{P} := \bigcup_{r\ge 1} \mathcal{P}_r.$$

Note at this point that the map $\rho : \mathcal{P} \to \mathcal{P},\ \mathbf{p} \mapsto \mathbf{p}'$ is an involution, and the restriction to one of the other sets in Definition 21 is also an involution. By abuse of notation, we also denote the restricted maps by ρ.

Theorem 18 ([12, Theorem 4.6]). *For every $\zeta_q(k_1, \dots, k_r; Q_1, \dots, Q_r) \in \mathcal{Z}_q$ with $r \in \mathbb{N}$, there are rational numbers $a_{\mathbf{p}} \in \mathbb{Q}$ for all partitions $\mathbf{p} \in \mathcal{P}_{\le r}$ such that*

$$\zeta_q(k_1, \dots, k_r; Q_1, \dots, Q_r) = \sum_{((m_1,\dots,m_{r'}),(n_1,\dots,n_{r'}))\in\mathcal{P}_{\le r}} a_{\substack{m_1,\dots,m_{r'}\\ n_1,\dots,n_{r'}}} q^{m_1 n_1 + \cdots + m_{r'} n_{r'}}.$$

Moreover, these are polynomials in $m_1, \dots, m_{r'}, n_1, \dots, n_{r'}$.

Proof. The theorem follows from the expansion

$$\frac{q^{m\ell}}{(1-q^m)^k} = q^{m\ell} \sum_{n\ge 0} \binom{n+k-1}{k-1} q^{mn} = \sum_{n\ge \ell} \binom{n-\ell+k-1}{k-1} q^{mn}$$

for all $m \ge 1$, $0 \le \ell \le k$, and from $\deg(Q_j) \le k_j$, that all Q_j have rational coefficients and from the fact that binomial coefficients, in particular, are rational numbers too. The fact that the coefficients are polynomial in $m_1, \dots, m_{r'}, n_1, \dots, n_{r'}$ also follows directly from this expansion. ☐

Theorem 18 implies that for every $S \in \mathcal{Z}_q$ there is a map $\mathbf{a} : \mathcal{P} \to \mathbb{Q}$ and a rational number a_0 such that

$$S = a_0 + \sum_{N \geq 1} \left(\sum_{\mathbf{p} \in \mathcal{P}(N)} \mathbf{a}(\mathbf{p}) \right) q^N$$

and all but finitely many of the projections $\mathbf{a}_r := \mathbf{a}|_{\mathcal{P}_r}$, $r \geq 1$, are constant zero.

The mappings $\mathbf{a}$ do not have to be unique, but we can find a polynomial one for each element $S \in \mathcal{Z}_q$.

Theorem 19 ([12, Theorem 4.7])**.** *A q-series S is in $\mathcal{Z}_q$ if and only if there exists $f = (f_r)_{r \geq 0}$ with $f_r \in \mathbb{Q}[X_1, \ldots, X_r, Y_1, \ldots, Y_r]$ for $r \geq 1$ and $f_0 \in \mathbb{Q}$ such that*

(i) *$f_r \equiv 0$ for all but finite many r,*

(ii) $S = f_0 + \sum_{N \geq 1} \left(\sum_{r \geq 1} \sum_{(\boldsymbol{m}, \boldsymbol{n}) \in \mathcal{P}_r(N)} f_r(m_1, \ldots, m_r, n_1, \ldots, n_r) \right) q^N.$

Proof. Note first that for every bi-bracket, such an f exists by the original definition of bi-brackets,

$$\mathrm{g}\binom{k_1, \ldots, k_r}{d_1, \ldots, d_r} := \sum_{\substack{m_1 > \cdots > m_r > 0 \\ n_1, \ldots, n_r > 0}} \prod_{j=1}^{r} \frac{m_j^{d_j}}{d_j!} \frac{n_j^{k_j - 1}}{(k_j - 1)!} q^{m_j n_j}.$$

There, $f_s \equiv 0$ for all s except $s = r$, the depth of the bi-bracket. Moreover, f_r is a monomial (up to a rational factor) in $\mathbb{Q}[X_1, \ldots, Y_r]$. Indeed, since $d_i \geq 0$, $k_j \geq 1$ can take all values, the set of all f_r's coming from a bi-bracket forms a basis of $\mathbb{Q}[X_1, \ldots, Y_r]$. Furthermore, this holds for every $r \geq 1$.

Now, if $S \in \mathcal{Z}_q$, S is a rational linear combination of bi-brackets since they span $\mathcal{Z}_q$. In this case, S is of the desired shape since a possible f is a finite rational linear combination of monomials by the above remark. Hence, in particular, it is a polynomial again.

Conversely, suppose S is of the shape in the theorem. In that case, the monomials occurring in f correspond to bi-brackets as remarked, i.e., S is a rational linear combination of bi-brackets, hence an element of $\mathcal{Z}_q$. ☐

Example 2. By some straightforward calculation, we get

$$\zeta_q(1, 0, 2; X, 1, 1 + X) = \sum_{m_1 > m_2 > m_3 > 0} \frac{q^{m_1}}{1 - q^{m_1}} \frac{1 + q^{m_3}}{(1 - q^{m_3})^2}$$

$$
\begin{aligned}
&= 2 \sum_{\substack{m_1>m_2>0\\ n_1,n_2>0}} (m_1-m_2-1)(n_2+1)q^{m_1n_1+m_2n_2}\\
&\quad + \sum_{\substack{m_1>0\\ n_1>0}} \frac{(m_1-2)(m_1-1)}{2} q^{m_1n_1}.
\end{aligned}
$$

In the terminology of maps $\mathbf{a} : \mathcal{P} \to \mathbb{Q}$ we find now for $\zeta_q(1,0,2;X,1,1+X)$ a suitable map

$$
\begin{aligned}
&\mathbf{a} : \mathcal{P} \longrightarrow \mathbb{Q},\\
&(\mathbf{m},\mathbf{n}) \longmapsto \delta_{r=1}\frac{(m_1-2)(m_1-1)}{2} + \delta_{r=2}\cdot 2(m_1-m_2-1)(n_2+1).
\end{aligned}
$$

Especially, we see that Theorem 19 applies here with the polynomials

$$
\begin{aligned}
f_1(m_1,n_1) :=& \frac{(m_1-2)(m_1-1)}{2},\\
f_2(m_1,m_2,n_1,n_2) :=& 2(m_1-m_2-1)(n_2+1), \quad f_r :\equiv 0 \ (r>2).
\end{aligned}
$$

Remark 18. Functions like $\mathbf{a}$ directly connect qMZVs to so-called q-brackets. For a function $\mathbf{a} : \mathcal{P} \to \mathbb{Q}$, the q-bracket of $\mathbf{a}$ is defined as

$$
\langle \mathbf{a} \rangle_q := \frac{\sum\limits_{\lambda\in\mathcal{P}} \mathbf{a}(\lambda)q^{|\lambda|}}{\sum\limits_{\lambda\in\mathcal{P}} q^{|\lambda|}}.
$$

Bloch and Okounkov introduced them in [10] and they are of interest in current research since, under certain conditions on $\mathbf{a}$, the q-bracket is quasi-modular. Recall at this point that every quasi-modular form is, in particular, an element of $\mathcal{Z}_q$.

The connection between qMZVs and q-brackets will be described in [9]. For further research details on q-brackets, we refer to the works by Schneider [26], Zagier [32], and van Ittersum [31].

Lemma 1 ([12, Lemma 4.9])**.** *For all $r, N \geq 1$, and maps $\boldsymbol{a}_r : \mathcal{P}_r(N) \to \mathbb{Q}$ we have the equation*

$$
\sum_{\mathbf{p}\in\mathcal{P}_r(N)} \boldsymbol{a}_r(\mathbf{p}) = \sum_{\mathbf{p}\in\mathcal{P}_r(N)} \boldsymbol{a}_r(\rho(\mathbf{p})). \tag{4.1}
$$

Proof. The map ρ is an involution on $\mathcal{P}_r(N)$. □

This lemma is important when considering duality relations among qMZVs like Schlesinger–Zudilin duality. For details, see [12, Lemma 4.13].

4.1. Bi-brackets

For every bi-bracket $\mathrm{g}\binom{k_1,\ldots,k_r}{d_1,\ldots,d_r}$ the coefficient of q^N can be easily derived by the original definition.

$$\begin{aligned}\mathrm{g}\begin{pmatrix}k_1,\ldots,k_r\\d_1,\ldots,d_r\end{pmatrix} &= \sum_{\substack{m_1>\cdots>m_r>0\\n_1,\ldots,n_r>0}} \frac{m_1^{d_1}}{d_1!}\cdots\frac{m_r^{d_r}}{d_r!}\frac{n_1^{k_1-1}\ldots n_r^{k_r-1}}{(k_1-1)!\ldots(k_r-1)!}q^{m_1n_1+\cdots+m_rn_r}\\ &= \frac{1}{\prod\limits_{j=1} d_j!(k_j-1)!}\sum_{N>0}\left(\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_r(N)}\prod_{j=1}^r m_j^{d_j}n_j^{k_j-1}\right)q^N.\end{aligned}$$

Lemma 1 gives an explicit expression of the so-called *partition relation* [2, Equation (3.1)].

Lemma 2 ([12, Lemma 4.11])**.** *For all* $r\geq 1$*,* $d_1,\ldots,d_r\geq 0$*,* $k_1,\ldots,k_r\geq 1$*, we have*

$$\begin{aligned}\mathrm{g}\begin{pmatrix}k_1,\ldots,k_r\\d_1,\ldots,d_r\end{pmatrix} = &\sum_{\substack{0\leq k_i'\leq k_i-1\\ d_{i,j}'\geq 0,\,1\leq i\leq r,\\ 1\leq j\leq r-i+1\\ d_{i,1}'+\cdots+d_{i,r-i+1}'=d_i}}\prod_{j=1}^r\frac{(d_{1,j}'+\cdots+d_{r-j+1,j}')!(k_{r-j+1}'+k_{r-j+2}-1-k_{r-j+2}')!}{d_j!(k_j-1)!}\\ &\times\begin{pmatrix}d_j\\d_{j,1}',\ldots,d_{j,r-j+1}'\end{pmatrix}\begin{pmatrix}k_j-1\\k_j'\end{pmatrix}\\ &\times\mathrm{g}\begin{pmatrix}d_{1,1}'+\cdots+d_{1,r}',\ldots,d_{r-1,1}'+d_{r-1,2}',d_{r,1}'\\k_r'-k_{r+1}'-1+k_{r+1},\ldots,k_1'-k_2'-1+k_2\end{pmatrix},\end{aligned}$$

with $k_{r+1}:=k_{r+1}':=0$.

Interesting in the context of bi-brackets and Theorem 19 is the following refinement of Bachmann's conjecture [4, Conjecture 4.3] which says that brackets and bi-brackets span the same space.

Conjecture 6. Let P be a polynomial in $\mathbb{Q}[X_1,\ldots,X_r,Y_1,\ldots,Y_r]$ for some $r\geq 1$. Then there exists $Q=(Q_j)_{j\geq 1}$ with $Q_j\in\mathbb{Q}[X_1,\ldots,X_j]$ and $Q_j\equiv 0$ for all but finitely many j such that for every $N\geq 1$, with $M:=r+\sum\limits_{i=1}^r\deg_{Y_i}(P)$, we have

$$\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_r(N)}P(m_1,\ldots,m_r,n_1,\ldots,n_r)=\sum_{j=1}^M\sum_{(\mathbf{m},\mathbf{n})\in\mathcal{P}_j(N)}Q_j(n_1,\ldots,n_r).$$

4.2. SZ Model

Consider some SZ admissible index $\mathbf{k} = \left(k_1+1, \{0\}^{d_1}, \ldots, k_r+1, \{0\}^{d_r}\right)$ and observe

$$\zeta_q^{\mathrm{SZ}}(\mathbf{k}) = \sum_{\substack{m_1>\cdots>m_r>0\\ n_1,\ldots,n_r>0}} \prod_{j=1}^{r} \binom{m_j - m_{j+1} - 1}{d_j} \binom{n_j - 1}{k_j} q^{m_j n_j}.$$

We can interpret the coefficient of q^N as the number of partitions of N with rows and columns in the Young diagram marked as below.

Proposition 17 ([12, Proposition 4.14])**.** *The coefficient of q^N in $\zeta_q^{\mathrm{SZ}}(\mathbf{k})$ is the number of partitions of N with exactly r parts with markings in the Young diagram as follows: For all $1 \leq i \leq r$, there are d_i rows of the ith part without the last row marked. Furthermore, for all $1 \leq j \leq r$, there are k_j of the columns lying between the jth and $(j+1)$st rightmost corner of the Young diagram marked.*

Example 3. For $N = 126$, one of the marked partitions as described in Proposition 17 with exactly $r = 3$ parts and with $k_1 = 2$, $k_2 = k_3 = 1$, $d_1 = 2$, $d_2 = 1$, $d_3 = 3$, is the one of Figure 3.

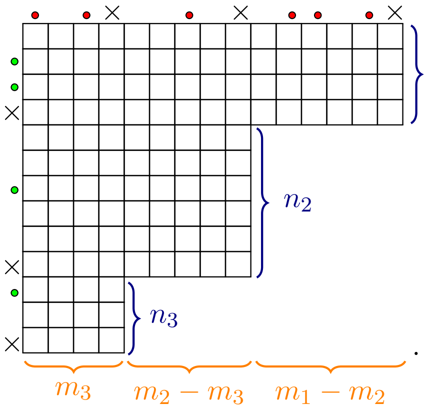


Figure 3: Marked partition mentioned in Example 3

The crosses × stand for the corresponding row/column not being allowed to be colored. That there is in every part a fixed row/column (we always select the lowest row/rightmost column) comes from the -1's in the binomial coefficients that we consider in the coefficient of q^N in $\zeta_q^{\mathrm{SZ}}(\mathbf{k})$.

Proof (of Proposition 17). Considering the index of the first sum of the coefficient of q^N in $\zeta_q^{\mathrm{SZ}}(\mathbf{k})$, we get that it is just the number of partitions of N with exactly r parts, every partition counted with the respective multiplicity, given as the product of binomial coefficients we have seen.

Now, given such a partition of N, the jth part consists of n_j rows, where there are $\binom{n_j-1}{k_j}$ ways of marking d_j of the rows of the jth part without the last one. Since those markings in every part are independent of the markings in the other parts, this row's coloring gives a multiplicity of

$$\binom{n_1-1}{k_1}\cdots\binom{n_r-1}{k_r}$$

of the given partition.

For the column markings, we use similar arguments. Since the jth part of the Young diagram has length m_j and the $(j+1)$st has length m_{j+1}, there are $m_j - m_{j+1} - 1$ columns between the jth and $(j+1)$st corner, counted from the right. Hence, with marking d_j of them, we get an additional factor $\binom{m_j-m_{j+1}-1}{d_j}$ for the multiplicity of the given partition (for every $1 \leq j \leq r$) and so exactly the coefficient of q^N of $\zeta_q^{\mathrm{SZ}}(\mathbf{k})$. □

Remark 19. By conjugating Young diagrams together with the markings, we get a new generating series of specific marked partitions. One can verify that this generating series is the SZ qMZV of the SZ dual index. Hence, using marked partitions, we deduce SZ duality [12, Section 4.2].

4.3. BZ Model

For admissible $\mathbf{k} = \big(k_1+1, \{1\}^{d_1-1}, \ldots, k_r+1, \{1\}^{d_r-1}\big)$, i.e., $k_j, d_j \geq 1$ for all $1 \leq j \leq r$, we compute

$$\zeta_q^{\mathrm{BZ}}(\mathbf{k}) = \sum_{\substack{m_1>n_1>\cdots>n_{d_1-1}>m_2>\cdots\\ >m_r>\cdots>n_{d_1+\cdots+d_r-r}>0\\ j_1,i_1,\ldots,i_{d_1-1},j_2\\ \ldots,i_{d_1+\cdots+d_r-r}\geq 0}} \binom{j_1}{k_1}\cdots\binom{j_r}{k_r} q^{m_1j_1+\cdots+m_rj_r+\sum_{\ell=1}^{d_1+\cdots+d_r-r} n_\ell i_\ell}.$$

The coefficient of q^N is again the number of partitions of N, where some of the rows and columns in the Young diagram are marked.

Proposition 18 ([12, Proposition 4.19])**.** *The coefficient of q^N in $\zeta_q^{\mathrm{BZ}}(\mathbf{k})$ corresponds to the number of partitions of N, where the corresponding Young diagram is split up into r sub-Young diagrams with at most $d_1, \ldots, d_r$ parts* *each. We mark k_j rows in the first part of the sub-Young diagram j for each $1 \leq j \leq r$. Furthermore, we mark all columns containing corners and some of the others such that the number of colored columns, only belonging to sub-Young diagram j, in total is d_j for each $1 \leq j \leq r$.*

Proof. We obtain the split up into r sub-Young diagrams by the first line of the sum index of the coefficient of q^N. Also, the row markings are self-explanatory when looking at the summand of our coefficient of q^N.

The marked columns represent the indices (from right to left) of shape j_ℓ and i_ℓ. If a marked column is not the rightmost one of a part, this corresponds to whether the corresponding multiplicity i_ℓ is zero. In this case, there is no n_ℓ-part in the partition. This is the reason for having exactly d_j marked columns that belong to sub-Young diagram j for each $1 \leq j \leq r$. Furthermore, it is the reason why we have at most (and not exactly) d_i parts in sub-Young diagram i. ☐

Example 4. The marked partition of $N = 118$, presented in Figure 4, has $r = 2$ sub-Young diagrams and is assigned to the index $\mathbf{k} = (3, 1, 1, 1, 3, 1, 1)$, i.e., $k_1 = k_2 = 2$, $d_1 = 4$, $d_2 = 3$.

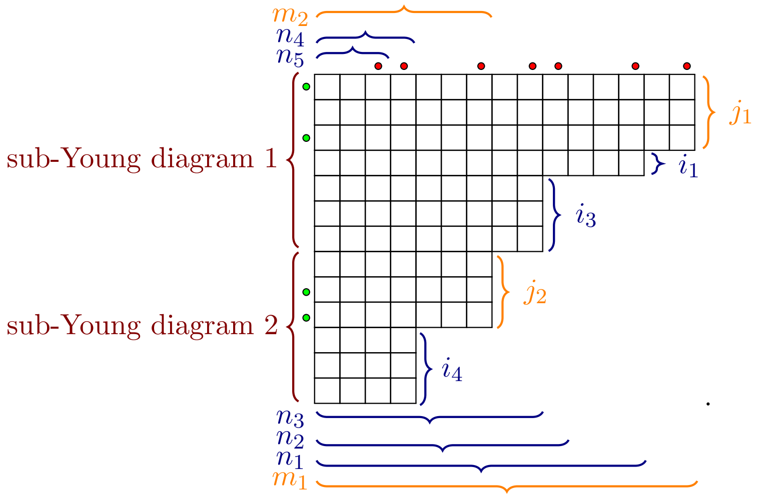


Figure 4: Marked partition of Example 4

Note that n_3 occurs with multiplicity $i_2 = 0$, and n_5 with multiplicity $i_5 = 0$, which is the reason that the columns corresponding to n_3 and n_5, respectively, are marked but contain no corner of the Young diagram.

One obtains Theorem 1, which is equivalent to BZ duality (see [12, Section 4.3]).

4.4. Partition Function, qMZVs and Conjugacy Classes

When studying particular SZ qMZVs, we get a connection to the partition numbers.

Lemma 3 ([12, Lemma 4.26])**.** *Let be p_N the number of partitions of N, then*

$$\sum_{r\geq 1} \zeta_q^{\mathrm{SZ}}(\{1\}^r) = \sum_{r\geq 1} \mathrm{g}\binom{\{1\}^r}{\{0\}^r} = \sum_{N\geq 1} p_N q^N.$$

Proof. The first equality is clear by the definition of bi-brackets. We now consider the left side first:

$$\begin{aligned}\sum_{r\geq 0} \zeta_q^{\mathrm{SZ}}(\{1\}^r) &= \sum_{r\geq 0}\sum_{m_1>\cdots>m_r>0} \frac{q^{m_1}}{1-q^{m_1}}\cdots\frac{q^{m_r}}{1-q^{m_r}} \\ &= \sum_{r\geq 0}\sum_{\substack{m_1>\cdots>m_r>0\\ n_1,\ldots,n_r>0}} q^{m_1n_1+\cdots+m_rn_r}.\end{aligned}$$

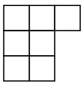
The coefficient of some q^N here is the sum over all $r \in \mathbb{N}_0$, where we sum the number of all partitions of N with exactly r different parts, i.e., the number of partitions of N, p_N. □

The partition function also occurs in contexts other than qMZVs, namely when considering equivalence classes of the symmetric group $\mathcal{S}_n$. We refer to [16, Section 4] for more details.

Lemma 4. *Partitions of $n \in \mathbb{N}$ and conjugacy classes of $\mathcal{S}_n$ are in 1:1 correspondence. In particular, the number of conjugacy classes of $\mathcal{S}_n$ is p_n.*

Proof. Write every $\sigma \in \mathcal{S}_n$ as a union of cycles. The lengths of the cycles form a partition of n. Since a conjugacy class $[\sigma]$ of $\mathcal{S}_n$ is uniquely determined by the lengths of cycles of σ - conjugacy means only to rename the elements $1, \ldots, n$, but not to change the structure of σ - the claim follows. □

Example 5. The conjugacy class of $\sigma = (1\,4\,3)(2\,6)(5\,7) \in \mathcal{S}_7$ corresponds to the partition



of 3+2+2=7.

Remark 20. Lemmas 3 and 4 give a remarkable connection between the number of conjugacy classes of $\mathcal{S}_n$ and SZ qMZVs. More precisely, fixing r and n, the coefficient of q^n in $\zeta_q^{\mathrm{SZ}}(\{1\}^r)$ is the number of conjugacy classes of $\mathcal{S}_n$ with cycles of exactly r different lengths.

This remark should be the motivation for the following theorem.

Theorem 20 ([12, Theorem 4.30])**.** *Let K be a finite field with c elements. Then we have*

$$\mathrm{G}_K := \sum_{n\geq 0} a_{n,K} q^n = \sum_{r\geq 0} (c-1)^r \zeta_q^{\mathrm{OOZ}}(\{1\}^r),$$

where $a_{n,K}$ is the number of conjugacy classes of $GL(n,K)$ with the convention $a_{0,K} := 1$ for every field K.

One can visualize the proof using marked partitions. For details and connected results about several representations of the numbers $a_{n,K}$, we refer to [12, Section 4.6].

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