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Uncertainty Principles from Representations of Lie Groups

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Outline

- Uncertainty Principles for Self/Skew adjoint operators
- Operators From Representations of Lie Groups
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- Uncertainty Principle on the Circle

Self/Skew Adjoint Operators

If A, B are unbounded then

$$|([A,B]x,x)| \leq 2\|Ax\|\|Bx\|,$$

for $x \in D(A) \cap D(B) \cap D([A,B])$.

It might happen that $D([A,B]) = \{0\}$

If $[A,B]$ is densely defined and closable we ask if this inequality extends to

$$|(\overline{[A,B]}x,x)| \leq 2\|Ax\|\|Bx\|,$$

for $x \in D(A) \cap D(B) \cap D(\overline{[A,B]})$.

Operators from Lie group representations

If G is a Lie group with Lie algebra $\mathfrak{g}$ and (π, H) is a unitary representation then

$$\phi(X)u = \lim_{t\to 0} \frac{\pi(\exp tX)u - u}{t}$$

defines a representation of $\mathfrak{g}$ on H_π^∞.
$\phi(X)$ is densely defined and skew-symmetric

$$(\phi(X)u, v) = \lim_{t\to 0} t^{-1}(\pi(\exp tX)u - u, v) = -(u, \phi(X)v)$$

So $\phi(X)$ is closable and we denote this closure $\pi(X)$. The domain of $\pi(X)$ is exactly the u for which the limit above exists.

Uncertainty by Kraus and Folland/Sitaram

It can be shown that $\overline{[\pi(X), \pi(Y)]} = \pi([X, Y])$.

[Kraus] showed that for $u \in D(\pi(X)) \cap D(\pi(Y)) \cap D(\pi([X, Y]))$

$$|(\pi([X, Y])u, u)| \leq 2\|\pi(X)u\|\|\pi(Y)u\|,$$

when G is 3-dimensional.

[Folland and Sitaram] proved (using a result by [Nelson]) that it is still true if $X, Y, [X, Y]$ span an ideal in $\mathfrak{g}$. They further asked the question whether the ideal restraint could be omitted.

We found a positive answer to this in [C] using a result by [Segal].

Main idea

With left Haar measure da define for $f \in C_c^\infty(G)$

$$\pi(f)u = \int_G f(a)\pi(a)u\,da$$

Show that for $u \in D(\pi(X))$

$$\pi(f)\pi(X)u - \pi(X)\pi(f)u = \pi(Lf + Rf + \gamma f)u$$

where

$$Lf(a) = \lim_{t\to 0} \frac{f(a\exp tX) - f(a)}{t}$$

$$Rf(a) = -\lim_{t\to 0} \frac{f(\exp tXa) - f(a)}{t}$$

$$\gamma = \lim_{t\to 0} \frac{\Delta(\exp tX) - 1}{t}$$

Main idea

It is a result of [Segal] that we can choose $f_n \to \delta$ such that $\|Lf_n + Rf_n + \gamma f_n\|_1$ is bounded and

$$\int_G Lf_n(a) + Rf_n(a) + \gamma f_n(a)\, da \to 0$$

such that (strong convergence)

$$\pi(f_n)\pi(X)u - \pi(X)\pi(f_n)u \to 0$$

from which we obtain

$$\pi(X)\pi(f_n)u \to \pi(X)u$$

The functions f_n are such that for local coordinates $x_1, \dots, x_n$

$$f_n(x) = \begin{cases} cn \prod_{i=1}^{n} \exp\left(\frac{1}{n^2 x_i^2 - 1}\right) & \text{if } |x_i| < 1/n \\ 0 & \text{elsewhere} \end{cases}$$

Heisenberg Uncertainty Principle

For the Heisenberg group

$$G = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \middle| x, y, z \in \mathbb{R} \right\}$$

with Lie algebra basis

$$X = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, [X, Y] = Z = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and representation on $L^2(\mathbb{R})$

$$\pi \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} f(t) = e^{iz+ty} f(t+x)$$

Heisenberg Uncertainty Principle

we get

$$\pi(X)f(t) = f'(t), \pi(Y) = itf(t), \pi(Z)f(t) = if(t)$$

and so we obtain the classical Heisenberg uncertainty principle

$$\Big(\int |f'(x)|^2 dx\Big)\Big(\int |xf(x)|^2 dx\Big) \geq \frac{1}{4}\|f\|_2^4$$

for $f(x), f'(x), xf(x)$ in $L^2(\mathbb{R})$.
We can obtain similar results for the n-dimensional Heisenberg group.

Euclidean Motion Group

Let $G = SO(2) \rtimes \mathbb{R}^2$ with unitary representation on $L^2(S^1)$

$$\pi(A, a)f(x) = e^{ib \cdot x} f(A^{-1}x)$$

Then for

$$X = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Y_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

we obtain the uncertainty principle

$$|(Sf, f)|^2 \leq 4\|Tf\|^2\|f\|^2$$

where

$$Sf(x) = -xf(x), Tf(x) = \frac{d}{dt}\Big|_{t=0} f\left(\begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix} x\right)$$

This is an uncertainty principle by Breitenberger.

References I

- Breitenberger: *Uncertainty measures and uncertainty relations for angle observables*
- C: *The uncertainty principle for operators determined by Lie groups*
- Folland and Sitaram: *The uncertainty principle: a mathematical survey.*
- Kraus: *A further remark on uncertainty relations*
- Nelson: *Analytic vectors*
- Segal: *A class of operator algebras which are determined by groups*