Sampling in reproducing kernel Banach spaces on Lie groups

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Overview

Idea: Some irregular sampling theorems for band limited functions use smoothness to obtain sampling results (for example Gröchenig and Pesenson). Extend these results to reproducing kernel Banach spaces on Lie groups.

Plan for talk:

- Classical irregular sampling results
- Reproducing kernel Banach spaces
- Smoothness of functions and sampling
- Smoothness of kernel and sampling
- Application to coorbit theory

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Band limited functions

Let $\mathcal{F}$ be the extension to $L^2(\mathbb{R}^n)$ of

$$\mathcal{F}f(w) = \frac{1}{(2\pi)^{n/2}} \int f(x) e^{-ix\cdot w}\, dx$$

Let $L^2_\Omega = \{f \in L^2 \cap C \mid \mathrm{supp}(\mathcal{F}f) \subseteq \Omega\}$ denote the space of Ω-band-limited functions.

Theorem (Gröchenig)

For an increasing sequence x_n without density points and with $\lim_{n\to\pm\infty} x_n = \pm\infty$ and $\delta := \sup(x_{n+1} - x_n) < \frac{\pi}{\omega}$ we have

$$\sum_n \frac{x_{n+1} - x_{n-1}}{2} |f(x_n)|^2 \sim \|f\|^2_{L^2}$$

Thus $\psi_n(x) = \sqrt{\frac{x_{n+1}-x_{n-1}}{2}}\psi(x - x_n)$ form a frame for L^2_Ω.

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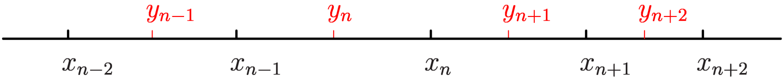
Proof of irregular sampling theorem by Gröchenig



The line is split such that $[y_n, y_{n+1}] \subseteq [x_n - \delta, x_n + \delta]$ and $\sum_n 1_{[y_n, y_{n+1}]} = 1$, i.e. we have a BUPU. The frame inequality follows if

$$\Big\| f - \sum_n f(x_n) 1_{[y_n, y_{n+1}]} \Big\| = \Big\| \sum_n |f - f(x_n)| 1_{[y_n, y_{n+1}]} \Big\| < \|f\|$$

Gröchenig uses that for $x \in [y_n, y_{n+1}]$

$$\begin{aligned} |f(x) - f(x_n)| &= |f(x) - f(x + t_n)| \\ &= \Big| \int_0^{t_n} f'(x+t)\, dt \Big| \\ &\leq \int_{-\delta}^{\delta} |f'(x+t)|\, dt \end{aligned}$$

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Reproducing kernel Banach spaces

Let G be a Lie group with left Haar measure dx. B is a solid Banach left and right invariant function space on G for which convergence in B implies convergence locally in measure. Denote the dual of B by B^*. Assume that $0 \neq \phi \in B \cap B^*$ satisfies

$$\phi * \phi(x) = \int \phi(y)\phi(y^{-1}x)\, dy = \phi(x)$$

then

$$B_\phi = \{f \in B \mid f = f * \phi\}$$

is a reproducing kernel Banach subspace of B.

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Smoothness of functions and sampling

As before (idea by Feichtinger and Gröchenig) we will investigate approximation of $f \in B_\phi$ by sums of the type

$$\sum_i f(x_i)\psi_i$$

where $0 \leq \psi_i \leq 1_{x_i U}$ is a partition of unity.
Fix a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ and define

$$U_\epsilon = \{e^{t_1 X_1} \cdots e^{t_n X_n} \mid -\epsilon \leq t_k \leq \epsilon\}$$

Let x_i be such that $x_i U_\epsilon$ have the finite covering property of G and find a partition of unity $0 \leq \psi_i \leq 1_{x_i U_\epsilon}$.

Smoothness of functions and sampling

Define right and left differentiation in the direction X as

$$R(X)f(x) = \frac{d}{dt}\Big|_{t=0} f(xe^{tX}) \qquad L(X)f(x) = \frac{d}{dt}\Big|_{t=0} f(e^{tX}x)$$

For $|\alpha| = m$ define

$$R^\alpha f = R_{X_{\alpha(1)}} R_{X_{\alpha(2)}} \cdots R_{X_{\alpha(m)}} f \qquad L^\alpha f = L_{X_{\alpha(1)}} L_{X_{\alpha(2)}} \cdots L_{X_{\alpha(m)}} f$$

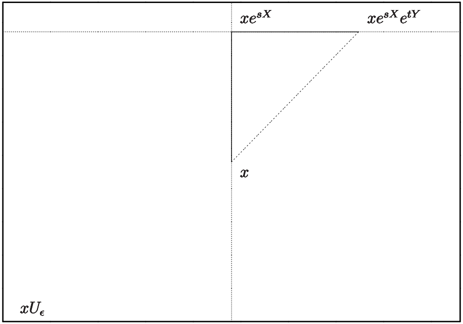
Lemma (C.)

If $f \in B$ is smooth with right derivatives in B then

$$\|f - \sum_i f(x_i)\psi_i\|_B \leq C_\epsilon \sum_{|\alpha| \leq \dim(G)} \|R^\alpha f\|_B$$

where $C_\epsilon \to 0$ as $\epsilon \to 0$.

Proof in two dimensions



$$\sup_{|s|,|t|\leq\epsilon} |f(x) - f(xe^{sX}e^{tY})| \leq \int_{-\epsilon}^{\epsilon} |R(X)f(xe^{rX})| + |R(Y)f(xe^{sX}e^{rY})| \, dr$$

$$\leq \int_{-\epsilon}^{\epsilon} |R(X)f(xe^{rX})| + |R(Y)f(xe^{rY})| + |R(Y)f(xe^{rY}e^{sAd_{rY}(X)}) - R(Y)f(xe^{rY})| \, dr$$

Smoothness of the kernel

Theorem (C.)

If $B \ni f \mapsto f * |R^\alpha \phi| \in B$ is continuous for all $|\alpha| \leq \dim(G)$ then

$$T_1 f = \sum_i f(x_i)\psi_i * \phi$$

is invertible on B_ϕ if x_i are close enough.

We can also discretize the reproducing formula $f = f * \phi$:

Theorem (C.)

If $B \ni f \mapsto f * |L^\alpha \phi| \in B$ and $B \ni f \mapsto f * |R^\alpha \phi| \in B$ are continuous for $|\alpha| \leq \dim(G)$ then with $c_i = \int \psi_i$

$$T_2 f = \sum_i c_i f(x_i)\ell_{x_i}\phi$$

is invertible on B_ϕ when x_i are close enough.

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Coorbits

Let π be a representation of G on a Fréchet space S which is weakly dense in its conjugate dual S^*. For a non-zero $u \in S$ define the wavelet transform $W_u(v)(x) = \langle v, \pi(x)u \rangle$

Theorem (C. and Ólafsson)

If

$$W_u(v) * W_u(u) = W_u(v) \qquad \text{for all } v \in S^*, \text{ and} \tag{1}$$

$$B \times S \ni (F, v) \mapsto \int F(x) W_u(v)(x^{-1})\, dx \in \mathbb{C} \text{ is continuous} \tag{2}$$

then

$$\mathrm{Co}_S^u B = \{v \in S^* \mid W_u(v) \in B\}$$

is a Banach space isometrically isomorphic to the reproducing kernel Banach space B_ϕ with $\phi(x) = \langle u, \pi(x)u \rangle$.

Examples

- Band limited functions on both $\mathbb{R}^n$ and on homogeneous spaces $X = G/K$ where (G, K) Gelfand pair.
- Homogeneous Besov spaces on $\mathbb{R}^n$, stratified Lie groups (Führ,Geller,Mayeli) and symmetric cones(?) (Bekolle, Bonami, Garrigos, Ricci)
- Bergman spaces on upper half plane and other tube type domains? (Bekolle, Bonami, Garrigos, Ricci)
- Modulation spaces by Feichtinger (model spaces for coorbits)
- Original coorbits by Feicthinger and Gröchenig for integrable representations

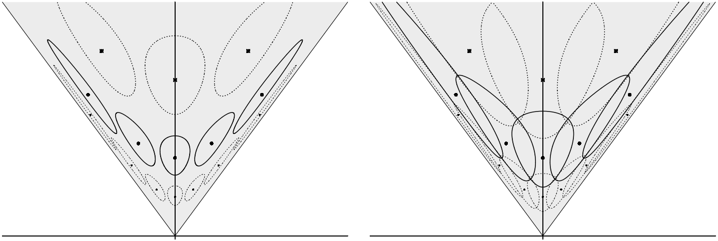
Example: Besov spaces on light cones

Λ is the forward light cone in $\mathbb{R}^n$ and

$$\mathcal{S}_\Lambda = \{f \in \mathcal{S}(\mathbb{R}^n) \mid \operatorname{supp}\widehat{f} \subseteq \Lambda\}.$$

A Whitney cover is a collection of translates of a ball such that

$$x_j B_{r/2}(e) \quad \text{disjoint and} \quad \Lambda \subseteq x_j B_r(e)$$



Example: Besov spaces on light cones

Let ψ_j be a Littlewood-Paley decomposition, satisfying $\mathrm{supp}\widehat{\psi}_j \subseteq x_j B_r(e)$ and $\sum_j \widehat{\psi}_j = 1_\Lambda$.

For $1 \leq p, q < \infty$ define the norm

$$\|f\|_{B_s^{p,q}} = \Big(\sum_j \det(w_j)^{-s} \|f * \psi_j\|_p^q\Big)^{1/q}$$

and the space $B_s^{p,q} = \{f \in \mathcal{S}'_\Lambda \mid \|f\|_{B_s^{p,q}} < \infty\}$.

Theorem

$B_s^{p,q}$ are coorbits for the quasiregular representation of $G = \mathbb{R}_+ SO_0(n-1,1) \rtimes \mathbb{R}^n$.

Coorbits, Gårding vectors and atomic decompositions

Let (π, H) be a square integrable representation and (S, H, S^*) a Gelfand triple such that (1) and (2) are satisfied for some u. Let $\widetilde{u} = \int g(x)\pi(x)u\,dx$ be a non-zero Gårding vector.

Theorem (C.)

If $B \ni F \mapsto F * |W_u(u)| \in B$ is continuous then $\mathrm{Co}_S^u B = \mathrm{Co}_S^{\widetilde{u}} B$. Further there is a sequence space B_d and $\lambda_i \in (\mathrm{Co}_S^{\widetilde{u}} B)^*$ such that for any $f \in \mathrm{Co}_S^{\widetilde{u}} B$ and x_i close enough

1. $\|\{\lambda_i(f)\}\|_{B_d} \sim \|f\|_{\mathrm{CoB}}$
2. $f = \sum_i \lambda_i(f)\pi(x_i)\widetilde{u}$

Proof:Since $\phi(x) = W_{\widetilde{u}}(\widetilde{u}) = g * W_u(u) * g^*$ all the derivatives of ϕ satisfy the sampling theorems.