# Besov spaces on stratified Lie groups and atomic decompositon through representation theory

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joint work with Azita Mayeli and Gestur Ólafsson

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Homogeneous Besov space norm is ([Peetre] and [Triebel])

$$\|f\|_{B_s^{p,q}} := \Big(\sum_j 2^{-sjq}\|\mathcal{F}^{-1}\widehat{\varphi}_j\mathcal{F}(f)\|_{L^p}^q\Big)^{!/q} = \Big(\sum_j 2^{-sjq}\|f * \varphi_j\|_{L^p}^q\Big)^{1/q}.$$

For Schwartz function $\psi$ define the wavelet transform of a tempered distribution $f$ by

$$W_\psi(f)(a,b) := \langle f, \pi(a,b)\psi\rangle = \frac{1}{a^{n/2}} \int f(x)\psi\Big(\frac{x-b}{a}\Big)\, dx.$$

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Triebel and Feichtinger/Gröchenig:

$$\begin{aligned}\|f\|_{B_s^{p,q}} &\sim \int \Big( \int |W_\varphi(f)(a,b)|^p\,db\Big)^{q/p} a^{s'}\,da \\ &\sim \sum_i a_i^{s'} \Big( \sum_j |W_\varphi(f)(a_i,b_{ij})|^p\Big)^{q/p}\end{aligned}$$

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For Feichtinger/Gröchenig an irredicible representation is needed (replace $a$ by $aK$ where $K \in \mathrm{SO}(n)$), but radial functions are cyclic.

# Heisenberg group

The $2n+1$-dimensional Heisenberg group can be realized as

$$\mathbb{H}_n = \{(z,t) \mid z = x + iy \in \mathbb{C}^n, t \in \mathbb{R}\}$$

with composition

$$(z_1,t_1)(z_2,t_2) = (z_1 + z_2, t_1 + t_2 + \frac{1}{2}\mathrm{Im}(z_1 \cdot \overline{z_2})).$$

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The Lie algebra is the set

$$\mathfrak{h}_n = \{(x_1X_1 \cdots x_nX_n + i(y_1Y_1 \cdots y_nY_n), tT) \mid x_k, y_k, t \in \mathbb{R}\}$$

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There is a homogeneous norm $|(z,t)| = \sqrt[4]{|z|^4 + t^2}$ satisfying

$$|a(z,t)| = a|(z,t)|$$

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- $\pi$ not irreducible on $L^2(G)$ or $S_0(G)$, but there are cyclic vectors [Führ/Mayeli].

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Similar results using heat kernels have been obtained by Führ and Mayeli.

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- This implies continuities needed for atomic decompositions.