Sampling in reproducing kernel Banach spaces on Lie groups

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Norbert Wiener Center for Harmonic Analysis and Applications

Overview

Idea: Some irregular sampling theorems for band limited functions use smoothness to obtain sampling results (for example Gröchenig and Pesenson). Extend these results to reproducing kernel Banach spaces on Lie groups.

Plan for talk:

- Reproducing kernel Banach spaces
- Smoothness of functions and sampling
- Smoothness of kernel and sampling
- Application to coorbit theory

Reproducing kernel Banach spaces

Let G be a Lie group with left Haar measure dx. B is a solid Banach function space on G for which convergence in B implies convergence locally in measure. Denote the dual of B by B^*. Assume that $0 \neq \phi \in B \cap B^*$ satisfies

$$\phi * \phi(x) = \int \phi(y)\phi(y^{-1}x)\, dy = \phi(x)$$

then

$$B_\phi = \{f \in B \mid f = f * \phi\}$$

is a reproducing kernel Banach subspace of B.

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We will investigate approximation of $f \in B_\phi$ by sums of the type

$$\sum_i f(x_i)\psi_i$$

where $0 \leq \psi_i \leq 1_{x_i U}$ is a partition of unity.
Fix a basis $X_1, \ldots, X_n$ for $\mathfrak{g}$ and define

$$U_\epsilon = \{e^{t_1 X_1} \cdots e^{t_n X_n} \mid -\epsilon \leq t_k \leq \epsilon\}$$

Let x_i be such that $x_i U_\epsilon$ have the finite covering property of G and find a partition of unity $0 \leq \psi_i \leq 1_{x_i U_\epsilon}$.

Smoothness of functions and sampling

Define right and left differentiation in the direction X as

$$R(X)f(x) = \frac{d}{dt}\Big|_{t=0} f(xe^{tX}) \qquad L(X)f(x) = \frac{d}{dt}\Big|_{t=0} f(e^{tX}x)$$

For $|\alpha| = m$ define

$$R^\alpha f = R_{X_{\alpha(1)}} R_{X_{\alpha(2)}} \cdots R_{X_{\alpha(m)}} f \qquad L^\alpha f = L_{X_{\alpha(1)}} L_{X_{\alpha(2)}} \cdots L_{X_{\alpha(m)}} f$$

Lemma: If $f \in B$ is smooth with right derivatives in B then

$$\|f - \sum_i f(x_i)\psi_i\|_B \leq C_\epsilon \sum_{|\alpha| \leq \dim(G)} \|R^\alpha f\|_B$$

where $C_\epsilon \to 0$ as $\epsilon \to 0$.

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Proof for $(ax+b)$-group

Let X_1 and X_2 be such that

$$e^{tX_1} = \begin{pmatrix} e^t & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad e^{tX_2} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

Then for $x \in x_i U_\epsilon$ we have

$$\begin{aligned} |f(x) - f(x_i)| &= |f(x) - f(xe^{s_2 X_2} e^{s_1 X_1})| \\ &\leq |f(x) - f(xe^{s_2 X_2})| + |f(xe^{s_2 X_2}) - f(xe^{s_2 X_2} e^{s_1 X_1})| \\ &\leq \int_{-\epsilon}^{\epsilon} |R(X_2) f(xe^{tX_2})| \, dt + \int_{-\epsilon}^{\epsilon} |R(X_1) f(xe^{s_2 X_2} e^{tX_1})| \, dt \\ &= \int_{-\epsilon}^{\epsilon} |R(X_2) f(xe^{tX_2})| \, dt + \int_{-\epsilon}^{\epsilon} |R(X_1) f(xe^{tX_1} e^{s_2 e^{-t} X_2})| \, dt, \end{aligned}$$

since $e^{s_2 X_2} e^{tX_1} = e^{tX_1} e^{s_2 e^{-t} X_2}$ or $Ad_{e^{tX_1}}(X_2) = e^{-t} X_2$.

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Proof for $(ax+b)$-group

$$
\begin{aligned}
&|R(X_1)f(xe^{s_2X_2}e^{tX_1})| \\
&\qquad \le |R(X_1)f(xe^{tX_1}e^{s_2e^{-t}X_2}) - R(X_1)f(xe^{tX_1})| + |R(X_1)f(xe^{tX_1})| \\
&\qquad \le \int_{-\epsilon}^{\epsilon} e^{-t}|R(X_2)R(X_1)f(xe^{tX_1}e^{se^{-t}X_2})|\,ds + |R(X_1)f(xe^{tX_1})|.
\end{aligned}
$$

We finally get

$$
\begin{aligned}
|f(x)-f(x_i)| \le &\int_{-\epsilon}^{\epsilon} |R(X_2)f(xe^{tX_2})|\,dt \\
&+ \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} e^{-t}|R(X_2)R(X_1)f(xe^{sX_2}e^{tX_1})|\,dt\,ds \\
&+ \int_{-\epsilon}^{\epsilon} |R(X_1)f(xe^{tX_1})|\,dt
\end{aligned}
$$

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Smoothness of functions and sampling

If convolution with ϕ is continuous we have

$$\|f - \sum_i f(x_i)\psi_i * \phi\|_{B_\phi} \leq C_\epsilon \sum_{|\alpha|\leq(dim)(G)} \|R^\alpha f\|_B$$

where $C_\epsilon \to 0$ as $\epsilon \to 0$.
Theorem: If right differentiation $B_\phi \to B$ is continuous then by choosing ϵ small enough the operator

$$T_1 f = \sum_i f(x_i)\psi_i * \phi$$

becomes invertible on B_ϕ.

Smoothness of the kernel

Evaluating the local oscillations of the kernel we can obtain sampling theorems involving derivatives of the kernel.

Theorem: If $B \ni f \mapsto f * |R^\alpha \phi| \in B$ is continuous for $|\alpha| \leq \dim(G)$ then we can choose points x_i close enough that T_1 is invertible on B_ϕ.

By similar estimates we can discretize the reproducing formula $f = f * \phi$

Theorem: If $B \ni f \mapsto f * |L^\alpha \phi| \in B$ and $B \ni f \mapsto f * |R^\alpha \phi| \in B$ are continuous for $|\alpha| \leq \dim(G)$ then we can choose points x_i close enough that the following operator is invertible on B_ϕ.

$$T_2 f = \sum_i c_i f(x_i) \ell_{x_i} \phi$$

where $c_i = \int \psi_i$

Let π be a representation of G on a Fréchet space S which is weakly dense in its conjugate dual S^*. For a non-zero $u \in S$ define the wavelet transform

$$W_u(v)(x) = \langle v, \pi(x)u \rangle$$

If

$$W_u(v) * W_u(u) = W_u(v) \qquad \text{for all } v \in S^*, \text{ and} \tag{1}$$

$$B \times S \ni (F, v) \mapsto \int F(x) W_u(v)(x^{-1})\, dx \in \mathbb{C} \text{ is continuous} \tag{2}$$

then

$$\mathrm{Co}_S^u B = \{ v \in S^* \mid W_u(v) \in B \}$$

is a Banach space isometrically isomorphic to the reproducing kernel Banach space B_ϕ with $\phi(x) = \langle u, \pi(x)u \rangle$.

Coorbits on Lie groups for square integrable representations

Let (π, H) be a square integrable representation and (S, H, S^*) a Gelfand triple such that (1) and (2) are satisfied.
If $B \ni F \mapsto F * |W_u(u)| \in B$ is continuous, then any non-zero Gårding vector $\pi(g)u = \int g(x)\pi(x)u\,dx$ for $g \in C_c^\infty(G)$ defines the same coorbit space (norm equivalence)

$$\mathrm{Co}_S^u B = \mathrm{Co}_S^{\pi(g)u} B$$

Since $\phi(x) = W_{\pi(g)u}(\pi(g)u) = g * W_u(u) * g^*$ all the derivatives of ϕ satisfy the sampling theorems.

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Theorem: If $B \ni f \mapsto F * |W_u(u)| \in B$ is continuous, then for any Gårding vector $\widetilde{u}$ it holds that $\psi \in \mathrm{Co}_S^{\widetilde{u}} B$ can be reconstructed from the samples $W_{\widetilde{u}}(\psi)(x_i)$ if x_i are chosen close enough.

In particular the operators $T_k : B_{\widetilde{u}} \to B_{\widetilde{u}}$ are invertible.

Thus $\pi(x_i)\widetilde{u}$ provides both a frame and an atomic decomposition for $\mathrm{Co}_S^{\widetilde{u}} B$ (and $\mathrm{Co}_S^{u} B$).