# Coorbit spaces and discretizations

Jens Gerlach Christensen

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# Bergman spaces

Let $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and define the Bergman spaces

$$A^{p,\alpha} = \{f \in \mathcal{O}(D) | \int_D |f(z)|^p (1-|z|^2)^{\alpha-2} dz < \infty\}$$

We will now look at a different characterization of some of these Bergman spaces using square integrable representations.

We will be interested only in the case of a non-integrable representation, since the other case has already been covered by Feichtinger and Gröchenig.

We will also see some discretization results for these spaces.

# Bergman spaces

The affine group is $\mathbb{R}_+ \times \mathbb{R}$ can be regarded as the subgroup

$$G = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} a + a^{-1} + ib & b + i(a - a^{-1}) \\ b - i(a - a^{-1}) & a + a^{-1} - ib \end{pmatrix} \right\} \subseteq SU(1,1)$$

for $a > 0$ and $b \in \mathbb{R}$.

$G$ can be represented on the Hilbert space $H = A^{2,2}$ by

$$\pi \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} f(z) = (-\beta z + \bar{\alpha})^{-2} f\left( \frac{\alpha z - \bar{\beta}}{-\beta z + \bar{\alpha}} \right)$$

# Bergman spaces

The smooth vectors for this representation have been characterized by [Olafsson,Ørsted]

$$H_\pi^\infty = \left\{ \sum_k a_k z^k \middle| \forall m : \sum_k |a_k|^2 \frac{2^{2m} k^{4m}}{k+1} < \infty \right\}$$

with (conjugate) dual

$$H_\pi^{-\infty} = \left\{ \sum_k a_k z^k \middle| \exists m : \sum_k |a_k|^2 \frac{2^{-2m} k^{-4m}}{k+1} < \infty \right\}$$

Denote the conjugate dual pairing by $\langle \cdot, \cdot \rangle$ and define $V_1(f)(a,b) = \langle f, \pi(a,b)1 \rangle$.

# Bergman spaces

It can be shown that

$$A^{p,p} = \{f \in \mathcal{O}(D) | \int_D |f(z)|^p (1 - |z|^2)^{p-2} dz < \infty\}$$

can also be described by

$$A^{p,p} = \{f \in H_\pi^{-\infty} | V(f) \in L^p\}$$

with equivalent norm $\|f\| = \|V(f)\|_{L^p}$ for $p > 1$.

**Note:** $V_1(1) \notin L^1$, but $L^p * |V_1(1)| \subseteq L^p$.

# Notation

- Let $S$ be a Frechét space which is weakly dense in its conjugate dual $S^*$ and let $\langle v', u \rangle$ denote the conjugate dual pairing of $u \in S$ and $v' \in S^*$
- Let $\pi$ be a representation of a (Lie) group on $S$ and define the voice transform $V_u(v')(x) = \langle v', \pi(x)u \rangle$
- Denote by $\pi$ also the contragradient representation on $S^*$, i.e. $\langle \pi(x)v', u \rangle = \left\langle v', \pi(x^{-1})u \right\rangle$
- Let $Y$ be some left- and right-invariant (quasi) Banach Function space.

# The generalization (Coorbit spaces)

Assume there is a cyclic $u \in S$ such that

R1. $V_u(v) * V_u(u) = V_u(v)$ for all $v \in S$

R2. $Y * V_u(u) \subseteq Y$ and $f \mapsto f * V_u(u)$ is continuous

R3. If $f = f * V_u(u) \in Y$ then $v \mapsto \int f(x) \langle \pi(x)u, v \rangle \, dx$ is in $S^*$

R4. $S^* \ni v' \mapsto \int \langle \pi(x)u, u \rangle \langle v', \pi(x)u \rangle \, dx$ is continuous

and then define

$$\mathrm{Co}_S^u Y = \{ v' \in S^* | V_u(v') \in Y \}$$

with norm $\|v'\| = \|V_u(v')\|_Y$

## Theorem

- $\mathrm{Co}_S^u Y$ is a $\pi$-invariant Banach space
- $V_u : \mathrm{Co}_S^u Y \to Y * V_u(u) \subseteq Y$ is an isometric isomorphism

# Feichtinger/Gröchenig

Let $(\pi, H)$ be a unitary irreducible square integrable representation of $G$, and $w$ a submultiplicative weight on $G$.
Assume that $Y$ is such that

$$Y * L^1_w(G) \subseteq Y \qquad \text{and} \qquad \|F * f\|_Y \leq C\|F\|_Y\|f\|_{1,w}$$

Further assume there is a $u \neq 0$ s.t.

$$H^1_w = \{v \in H | V_u(v) \in L^1_w(G)\} \ni u$$

Then $H^1_w$ is a Banach space with norm $\|v\| = \|V_u(v)\|_{1,w}$, and (R1-R4) are satisfied with $S = H^1_w$. The coorbit space is then

$$\mathrm{Co}_{FG} Y = \{v \in (H^1_w)^* | V_u(v) \in Y\}$$

with norm $\|v\|_{FG} = \|V_u(v)\|_Y$.

# Sampling

Sampling can be done on $Y * V_u(u)$ instead of on $\mathrm{Co}Y$.

- $\{x_i\}$ is a countable subset of $G$
- $U$ is a compact neighbourhood of the identity for which $\#\{j | x_i U \cap x_j U \neq \emptyset\} < N$ for all $i$.
- $\{\psi_i\}$ is a partition of unity $\mathrm{supp}\psi_i \subseteq x_i U$

**Theorem**([Gröchenig]) If it is possible to pick $u$ and $U$ such that

$$V_u(u)^{\#} = \sup_{x \in U^{-1}} |r_x V_u(u) - V_u(u)| \in L^1_w$$

and $\|V_u(u)^{\#}\|_{L^1_w} < 1/\|V_u(u)\|_{L^1_w}$ then

$$Sf = \sum_i f(x_i)\psi_i * V_u(u)$$

is continuous with continuous inverse on $Y * V_u(u)$.

## Sampling in Bergman spaces

For the Bergman spaces it is still possible to show that $S$ is invertible even without the integrability condition.

**Lemma**
We can pick $U$ such that $V_1(1)^{\#} \leq C_U|V_1(1)|$ and as $U \to \{e\}$ we have $C_U \to 0$.

Using this lemma it can be shown that we can pick $U$ such that the operator

$$Sf = \sum_i f(x_i)\psi_i * V_1(1)$$

is continuous with continuous inverse on $L^p * V_1(1)$.

# Sampling in Bergman spaces

Similar results can be shown for

$$Tf = \sum_i c_i f(x_i) \ell_{x_i} V_1(1)$$

where $c_i = \int \psi_i(x) dx$.

The idea for the proof of this was found in [Zhu].

# Further projects

1. Coorbit spaces for homogeneous spaces and nilpotent Lie groups.
2. Bandlimited functions are also coorbits. The kernel involved ($\mathrm{sinc}$ function) is again not $L^1$. Sampling theorems exist, but we cannot prove estimates like in the lemma before. We would like discretization machinery to handle general coorbits.

# References I


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