INVARIANT MEASURES ON LOCALLY COMPACT GROUPS

JENS GERLACH CHRISTENSEN

Abstract. This is a survey about invariant integration on locally compact groups and its uses. The existence of a left invariant regular Borel measure on locally compact Hausdorff groups is proved. It is also proved that this measure is unique in some sense. A few examples of interesting locally compact groups are given.

Introduction

An important property of the Lebesgue measure on $\mathbb{R}^n$ is that it is translation invariant. This means that translation of a measurable set does not change the value of its measure.

Invariant measures are important tools in many areas of mathematics. For example the uncertainty principle related to Lie groups presented in [1] does not include any statements about an invariant measure, but the measure plays an important role in proving the theorem. On a Lie group we can construct an invariant measure from the Lebesgue measure on the Lie algebra. As will be shown in this paper the group structure also gives rise to an invariant measure. That these two measures are essentially the same then follows from the uniqueness statement also to be found in the present paper.

It is interesting to see how the Lebesgue measure on $\mathbb{R}^n$ can be generalised to groups.

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1. Locally compact groups

In this section we introduce the notion of locally compact groups and regular measures.

Definition 1.1. If G is a group and $\mathcal{T}$ is a topology on G such that $(x, y) \mapsto x^{-1} \cdot y$ is a continuous map from $G \times G \to G$ then G is called a *topological group*.

The topology $\mathcal{T}$ is *locally compact* if every open set contains a compact set with non-empty interior. It is called *Hausdorff* if for all $x \neq y$ there are disjoint open sets U and V such that $x \in U$ and $y \in V$.

Example 1.2. The following are examples of locally compact topological groups.

(1) Let $\mathcal{T}$ be the topology given by the usual metric $d(x,y) = |x-y|$ on $\mathbb{R}^n$, then $\mathbb{R}^n$ with addition is a locally compact Hausdorff group.

(2) Let $\mathcal{T}|_{\mathbb{R}_+}$ be the topology above restricted to $\mathbb{R}_+$ then $\mathbb{R}_+$ with multiplication is a locally compact Hausdorff group.

(3) The general linear group $\mathrm{GL}_n(\mathbb{R})$ with the topology it inherits from $\mathbb{R}^{n^2}$ is a locally compact toplogical group. The same holds for all subgroups of $\mathrm{GL}_n(\mathbb{R})$.

(4) The $(ax+b)$-group is the group of affine transformations of $\mathbb{R}$. It can be viewed as the subgroup of $\mathrm{GL}_2(\mathbb{R})$ given by

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} |a > 0 \text{ and } b \in \mathbb{R} \right\}$$

It is a group with many applications one of which is wavelet theory (for more references on this see [7]).

(5) The subgroup of $\mathrm{GL}_{n+2}(\mathbb{R})$ consisting of elements of the form

$$\begin{pmatrix} 1 & x^t & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y \in \mathbb{R}^n$ and $z \in \mathbb{R}$ is called the Heisenberg group. This group is used in harmonic analysis as can be seen in [4].

Notice that the first two groups are abelian while the last three groups are not abelian.

Now follows a basic separation result for locally compact Hausdorff groups, which will be used later in the text.

Theorem 1.3. If C and D are disjoint compact subsets of a locally compact Hausdorff group G then there are disjoint open sets U and V such that $C \subseteq U$ and $D \subseteq V$.

Proof. Let $x \in C$ and $y \in D$. Since G is Hausdorff there are disjoint open sets U and V such that $x \in U_{xy}$ and $y \in V_{xy}$. A finite amount of V_{xy} cover D so the finite intersection $U_x = \cap_y U_{xy}$ is open and contains x. U does not intersect $V_x = \cup_y V_{xy}$. Now pick a finite number of U_x to cover C. Then the set $U = \cup_x U_x$ does not intersect $V = \cap_x V_x$, and it holds that $C \subseteq U$ and $D \subseteq V$. $\square$

Now let us look at a few definitions regarding measures. I assume the reader has very basic knowledge of measure theory.

Definition 1.4. A Borel measure μ on a locally compact space is called *regular* when it holds that

(1) every compact set is μ-measurable;
(2) if A is measurable then $\mu(A) = \inf\{\mu(U) | A \subseteq U,\ U \text{ open}\}$;
(3) $\mu(U) = \sup\{\mu(C) | C \subseteq U,\ C \text{ compact}\}$ for each open set U.

Definition 1.5. A regular Borel measure μ on a locally compact group G is called a *left Haar measure* if

(1) μ is not the zero measure;
(2) the measure of a compact set is finite;
(3) for every $x \in G$ and all measurable sets E the left translate xE is measurable and $\mu(xE) = \mu(E)$.

2. Existence and uniqueness of invariant measure

From now on let G denote a locally compact Hausdorff group. In this section we show the existence of a left Haar measure on G. This proof follows the development in [6]. Haar first showed this result for second countable locally compact groups, and A. Weil later generalised to locally compact groups. The main result is the following

Theorem 2.1 (Main result)**.** For every locally compact Hausdorff group there exists a left Haar measure. If μ and ν are two left Haar measures on G then there is a $c > 0$ such that $\nu = c\mu$.

Let C be compact and N an open neighborhood of the identity e. Then $C \subseteq \cup_{x \in C} xN$ is an open covering of C. The set C is compact so a finite covering exists and we can therefore talk about a smallest such covering

(1) $\quad (C : N) = \text{the smallest amount of tranlates of } N \text{ covering } C$

Now let C_0 be a compact neighborhood of e, we then get a measure for how many copies of C_0 we need to cover C by

$$\tau_N(C) = \frac{(C : N)}{(C_0 : N)} \tag{2}$$

Lemma 2.2. Let C_0 be a compact neighborhood of e and N an open neighorhood of e. If C and D are compact then we have

(1) $\tau_N(xC) = \tau_N(C)$ for $x \in G$;
(2) $0 \leq \tau_N(C) < \infty$;
(3) if $C \subseteq D$ then $\tau_N(C) \leq \tau_N(D)$;
(4) $\tau_N(C \cup D) \leq \tau_N(C) + \tau_N(D)$;

(5) $\tau_N(C \cup D) = \tau_N(C) + \tau_N(D)$ if; $CN^{-1} \cap DN^{-1} = \emptyset$.

Apart from the last property τ_N is very close to being what is called a content (defined on p. 348 [8]).

Proof. The first property is due to the fact that a covering of C can be translated by x to cover xC. A covering of xC covers C after translation by x^{-1}. Thus $(xC : N) = (C : N)$. The second and third property are easily verified by the definition of τ_N. A covering of C and a covering of D together cover $C \cup D$ and so the fourth propery follows. If $CN^{-1} \cap DN^{-1} = \emptyset$ then $xN \cap D = \emptyset$ for $x \in C$. Thus no covering set of C can be used to cover D. The same holds for the covering sets of D. Therefore $(C : N) + (D : N) = (C \cup D : N)$ which proves the fifth and last property. □

The next step is to eliminate the use of the open set N. To do this we need the following

Lemma 2.3. Given a compact neighbourhood C_0 of e and a compact set C there is a compact interval $[0, k_C]$ such that $\tau_N(C) \in [0, k_C]$ for all neighborhoods N of e.

Proof. Let k_C be the smallest amount of translations of C_0° covering C. For every open neighbourhood N of e the amount of translates of N covering C is less than $k_C(C_0 : N)$. Therefore $(C : N) \leq k_C(C_0 : N)$ which proves the claim. □

This shows why it was important to define τ_N using the compact set C_0. From now on let C_0 be a fixed compact neighbourhood of e and let $\mathcal{K}$ denote the set of all compact subsets of G. By Tychonoffs theorem ([5, Theorem 37.3]) the product space

$$P = \Pi_{C \in \mathcal{K}}[0, k_C] \tag{3}$$

is compact in the product topology. Define the subset T_N of P by

$$T_N = \{\tau_M | M \subseteq N \text{ and } M \text{ is an open neighborhood of } e\}$$

If $M_1, \ldots, M_n$ are open neighborhoods of e then $M = \cap_{i=1}^n M_i$ is again open so $T_M \in \cap_{i=1}^n T_{M_i}$. Thus T_N is closed under finite intersections so $\bar{T}_N$ are closed sets which have the finite intersection property. Theorem 26.9 in [5] then gives that $\cap_N \bar{T}_N \neq \emptyset$. Choose a τ_0 in $\cap_N \bar{T}_N$.

Theorem 2.4. With τ_0 chosen as above and C and D compact it holds that

(1) $\tau_0(C \cup D) \leq \tau_0(C) + \tau_0(D)$
(2) $\tau_0(C \cup D) = \tau_0(C) + \tau_0(D)$ if $C \cap D = \emptyset$
(3) If $C \subseteq D$ then $\tau_0(C) \leq \tau_0(D)$

(4) $\tau_0(xC) = \tau_0(C)$ for every $x \in G$
(5) $\tau_0(C_0) = 1$

Proof. Let P be given as in (3). The topology on P is defined such that for every compact set C the mapping $f_C : P \to \mathbb{R}$ given by $f_C(\sigma) = \tau(C)$ is continuous. The set $A_1 = \{\sigma : \sigma(C \cup D) \leq \sigma(C) + \sigma(D), C, D \text{ compact}\}$ is closed since it is equal to $g^{-1}([0, \infty[)$ for $g = f_C + f_D - f_{C \cap D}$. By 4 in Lemma 2.2 it follows that $\tau_N \in A_1$ for every neighbourhood N of e. So $T_N \subseteq A_1$ for every N and since A_1 is closed $\bar{T}_N \subseteq A_1$. τ_0 is in $\bar{T}_N$ for all N and so $\tau_0 \in A_1$.

Let $A_2 = \{\sigma : \sigma(C \cup D) = \sigma(C) + \sigma(D) C, D \text{ disjoint } \}$ which is the closed set $g^{-1}(\{0\})$. For disjoint C and D there is an neighbourhood N_0 such that $CN_0^{-1} \cap DN_0^{-1} = \emptyset$. Thus $\bar{T}_{N_0} \subseteq A_2$ and $\tau_0 \in \bar{T}_{N_0}$.

Let $A_3 = \{\sigma : \sigma(C) \leq \sigma(D), C \subseteq D\}$ then $T_N \subseteq A_3$ for every N. A_3 is closed since it is $g^{-1}([0, \infty[)$ for $g = f_D - f_C$. So $\tau_0 \in \bar{T}_N \subseteq A_3$.

$A_4 = \{\sigma : \sigma(xC) = \sigma(C)\} = g^{-1}(\{0\})$ for $g = f_{xC} - f_C$ is closed. An argument similar to the previous shows that $\tau_0 \in A_4$.

$A_5 = \{\sigma : \sigma(C_0) = 1\} = g^{-1}(\{1\})$ for $g = f_{C_0}$. Note that $\tau_{C_0^\circ}(C_0) = 1$ so $\bar{T}_{C_0^\circ} \subseteq A_5$. Thus $\tau_0 \in A_5$. $\square$

To obtain a regular measure the following will be of great importance.

$$\tau(U) = \sup\{\tau_0(D) | D \subseteq U, D \text{ is compact}\} \tag{4}$$

Lemma 2.5. If U and V are open sets then $\tau(U \cup V) \subseteq \tau(U) + \tau(V)$.

Proof. Let $E \subseteq U \cup V$ be a compact. The sets $E \setminus U$ and $E \setminus V$ are disjoint and compact (they are closed subsets of E). Therefore by Theorem 1.3 there are open disjoint sets O_1 and O_2 such that $E \setminus U \subseteq O_1$ and $E \setminus V \subseteq O_2$. Then it holds that $E \setminus O_1 \subseteq U$ and $E \setminus O_2 \subseteq V$ and moreover $(E \setminus O_1) \cup (E \setminus O_2) = E$. From this it follows that

$$\tau_0(E) \leq \tau_0(E \setminus O_1) + \tau_0(E \setminus O_2) \leq \tau(U) + \tau(V).$$

This holds for all compact subsets E of $U \cup V$ and so the claim follows when we take the supremum over such E. $\square$

After this we can define an outer measure μ^* by

$$\mu^*(A) = \inf \left\{ \sum \tau(U_i) | A \subseteq \cup_{i=1}^{\infty} U_i, U_i \in \mathcal{T} \right\} \tag{5}$$

for every subset A of G.

A set B is called μ^*-measurable if

$$\mu^*(A) \geq \mu^*(A \cap B) + \mu^*(A \setminus B) \tag{6}$$

for all subsets A of G. The importance of μ^*-measurable sets is explained in the following theorem.

Theorem 2.6. The set function μ^* as defined above is an outer measure. The set M of μ^*-measurable sets is a σ-algebra and μ^* restricted to M is a measure in G.

Proof. Since $\emptyset$ is open it follows that $\mu^*(\emptyset) = \tau(\emptyset) = 0$. Also from the definition of τ it is easy to see that if $A \subseteq B$ then $\mu^*(A) \leq \mu^*(B)$ since a compact covering of B also covers A. μ^* is countably sub additive i.e.

$$\mu^*(\cup_{i=1}^\infty A_i) \leq \sum_{i=1}^\infty \mu^*(A_i),$$

since if U_i^j is a countable covering of A_i then $\cup_{i,j} U_i^j$ will also be a countable covering of $\cup_i A_i$.

The last claim of the theorem is called Caratheodory's Theorem which is the Theorem on page 289 in [8]. $\square$

Lemma 2.7. If C is compact then $\mu^*(C) \geq \tau_0(C)$.

Proof. Let $\cup_{i=1}^\infty U_i$ be an open cover of C and let $\cup_{j=1}^n U_{i_j}$ be a subcover. Since

$$\tau(\cup_{j=1}^n U_{i_j} = \sup\{\tau_0(D) | D \subseteq \cup_{j=1}^n U_{i_j} \text{where } D \text{ is compact }\}$$

it is true that $\tau_0(C) \leq \tau(\cup_{j=1}^n U_{i_j})$. From lemma 2.5 we get

$$\tau(\cup_{j=1}^n U_{i_j}) \leq \sum_{j=1}^n \tau(U_{i_j}) \leq \sum_i \tau(U_i)$$

and thus $\tau_0(C) \leq \mu^*(C)$. $\square$

Theorem 2.8. Every closed set is μ^*-measurable and μ^* restricted to the Borel sets is thus a regular Borel measure. The measure is left invariant Haar measure.

Proof. To show that every closed set F is μ^*-measurable we start by showing that

$$\mu^*(U) \geq \mu^*(F \setminus U) + \mu^*(F \cap U)$$

for all open sets U. Pick compact sets D and E such that $D \subseteq F \setminus U$ and $E \subseteq F \cap U$. Then D and E are disjoint and so

$$\mu^*(U) \geq \mu^*(D \cup E) \geq \tau_0(D \cup E) = \tau_0(D) + \tau_0(E)$$

Let D be given and take the supremum over compact E with $E \subseteq U \setminus D$ we get $\mu^*(U) \geq \tau_0(D) + \tau(U \setminus D)$ and since $U \setminus D$ is open and contains $U \cap F$ we have

$$\mu^*(U) \geq \tau_0(D) + \mu^*(U \cap F)$$

by the definition of μ^*. Next take the supremum over compact sets D with $D \subseteq U \setminus F$ to

$$\mu^*(U) \geq \tau(U \setminus F) + \mu^*(U \cap F) \geq \mu^*(U \setminus F) + \mu^*(U \cap F)$$

Now let A be any subset of G. If $\mu^*(A) = \infty$ the inequality (6) clearly holds. So assume that $\mu^*(A) < \infty$. For each $\epsilon > 0$ the definition of the outer measure ensures that there is an open covering $A \subseteq \cup_{i=1}^{\infty} U_i$ such that $\mu^*(A) + \epsilon > \sum_{i=1}^{\infty} \tau(U_i)$. Then it follows that

$$\begin{aligned}\mu^*(A) + \epsilon > \sum_{i=1}^{\infty} \tau(U_i) \geq \sum_{i=1}^{\infty} \mu^*(U_i) &\geq \sum_{i=1}^{\infty} \mu^*(U_i \setminus F) + \mu^*(U_i \cap F) \\ &\geq \mu^*((\cup_{i=1}^{\infty} U_i) \setminus F) + \mu^*((\cup_{i=1}^{\infty} U_i) \cap F) \\ &\geq \mu^*(A \setminus F) + \mu^*(A \cap F).\end{aligned}$$

This holds for any $\epsilon > 0$ which proves that F is μ^*-measurable.

Next I have to check that the restriction of μ^* to the Borel sets (σ-algebra generated by the closed sets) is a regular measure. Let A be μ-measurable with $\mu(A) < \infty$. For $\epsilon > 0$ choose a countable open covering $\cup_{i \in I} U_i$ such that $\mu(A) + \epsilon > \sum_{i \in I} \tau(U_i)$. The subadditivity of μ then gives $\mu(A) + \epsilon > \mu(\cup_{i \in I} U_i)$ and since $\cup_{i \in I} U_i$ is open it follows that $\mu(A) + \epsilon > \inf\{\mu(U) : U \text{ open and } A \subseteq U\}$. This shows the outer regularity of μ. If C is a compact subset of an open set U, then $\mu(C) \leq \mu(U)$ and so $\mu(U) \geq \sup\{\mu(D) : \text{compact } D \subseteq U\}$. Also $\tau_0(C) \leq \mu(C)$ and $\mu(U) \leq \tau(U)$ so

$$\begin{aligned}\mu(U) \leq \tau(U) &= \sup\{\tau_0(D) : \text{compact } D \subseteq U\} \\ &\leq \sup\{\mu(D) : \text{compact } D \subseteq U\}\end{aligned}$$

This shows that μ is inner regular.

Since $\mu(C_0) \geq \tau_0(C_0) = 1$ μ is not the zero measure. By the definition of τ_0 it follows that $\tau_0(C)$ is finite for all compact sets C. If D is a compact set then it can be covered by a finite number of translates $x_i D^\circ$. The set $C = \cup x_i D$ is compact and $D \subseteq \cup x_i D^\circ \subseteq C^\circ$. It then holds that

$$\mu(D) \leq \tau(C^\circ) \leq \tau_0(C) < \infty$$

where we use that $\tau(C^\circ) \leq \tau_0(C)$, which is true since for compact $E \subseteq C^\circ \subseteq C$ and then $\tau_0(E) \leq \tau_0(C)$.

The left invariance of μ follows from the left invariance of τ_0. □

Thus the existence of a left Haar measure has been proved for locally compact Hausdorff groups.

Remark 2.9. Assume that G is not Hausdorff and let $H = \overline{\{e\}}$. Then H is a closed normal subgroup of G and so G/H is a Hausdorff group and thus has a left Haar measure μ. Let $\kappa : G \to G/H$ be the canonical homomorphism. If we set $\bar{\mu}(A) = \mu(\kappa(A))$ for all Borel sets A then we obtain a left Haar measure $\bar{\mu}$ on G.

The following theorem gives the precise statement for uniqueness of a left Haar measure on G.

Theorem 2.10. Let G be a locally compact Hausdorff group and let μ and ν be left Haar measures on G. Then there is a $c > 0$ such that $\nu = c\mu$.

Proof. Let g be a non-zero function in $C_c(G)$ with $g(x) \geq 0$ for all $x \in G$ and let $f \in C_c(G)$.

First note that $\int g d\mu > 0$. Let U be a non-empty open set. By the definition of a left Haar measure there is a compact set C with positive measure. Let $\cup_{i=1}^{n} x_i U$ be a finite covering of C. Then $0 < \mu(K) \subseteq \sum_{i=1}^{n} \mu(x_i U) = n\mu(U)$ which shows that $\mu(U) > 0$. So since $g \geq 0$ is not the zero-function it follows that $\int g d\mu > 0$.

Also note that $\frac{\int f d\mu}{\int g d\mu}$ does not depend on the measure μ. This is seen by defining

$$h(x, y) = \frac{f(x)g(yx)}{\int g(zx)d\nu(z)}$$

which has compact support since $\int g(zx)d\nu(z) > 0$. Integration gives

$$\begin{aligned}
\int f(x)d\mu(x) &= \iint \frac{f(x)g(yx)}{\int g(zx)d\nu(z)} d\nu(y)d\mu(x) \\
&= \iint \frac{f(y^{-1}x)g(x)}{\int g(zy^{-1}x)d\nu(z)} d\mu(x)d\nu(y) \\
&= \iint \frac{f(y^{-1})g(x)}{\int g(zy^{-1})d\nu(z)} d\mu(x)d\nu(y) \\
&= \int g(x)d\mu(x) \int \frac{f(y^{-1})}{\int g(zy^{-1})d\nu(z)} d\nu(y)
\end{aligned}$$

The first equality is verified by calculation, the second follows from use of Fubini's theorem (see [8] p. 307) and the substitution $x \mapsto y^{-1}x$. The third equality is also due to Fubini's theorem and the substitution $y \mapsto xy$. It shows that the fraction $\frac{\int f d\mu}{\int g d\mu}$ does not depend on μ. So we can conclude that $\frac{\int f d\mu}{\int g d\mu} = \frac{\int f d\nu}{\int g d\nu}$. So for all $f \in C_c(G)$ it holds that $\int f d\nu = c \int f d\mu$. It can be shown that for regular measures μ the following is true $\mu(U) = \sup\{\int f d\mu : f \in C_c(G), 0 \leq f \leq 1_U\}$ for open sets U. See the proof of the Riesz-Markov theorem on page 352 in [8] for further details. Thus it follows that $\nu = c\mu$ on all open sets and thus on all Borel sets. $\square$

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Department of mathematics, Louisiana State University
E-mail address: `vepjan@math.lsu.edu`
URL: `http://www.math.lsu.edu/~vepjan`