CONVOLUTION ON L^p.

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1. Introduction.

This paper will look in depth at the L^p-conjecture and Young's inequality. It's based on an article from 1990 by Sadahiro Saeki. I chose to write it in english to get some practise in written english. Unfortunately the article is very thorough and precise, so it's been very hard to get my personal style through. It might sometimes look like I'm just retyping the article. But on to more important things.

The L^p-conjecture concerns general locally compact groups. It states that if $L^p(G)$ is closed under convolution for a $p \in]0, \infty[$, then G is compact. Young's inequality (1) is well known, but I haven't seen it before in my studies so I prove it here. The proof is taken from (20.18) in [2].

$$\|f * g\|_r \leq \|f\|_p \|g\|_q. \tag{1}$$

The last part looks at the sharpness of Young's inequality for locally compact abelian groups (LCA). The commutativity enables us to use some classifications of LCA-groups that will not be proved here. I investigate the indices p, q, r for which Young's inequality holds.

2. Introductory results.

Theorem 2.1. *If* $\inf \Delta(G) > 0$*, then* G *is unimodular.*

Proof. Assume G is not unimodular. Let $a \in G$ has $\Delta(a) < 1$. Such an a exists for if $\Delta(a) > 1$ (there exists an $\Delta(a) \neq 1$ since G is not unimodular) we can choose a^{-1}. But then $\Delta(a^n) \to 0$, which contradicts the fact that $\inf \Delta(G) > 0$. □

$$\sum_{k=2}^{\infty} \frac{1}{k \log^2 k} < \infty$$

since

$$\int_2^{\infty} \frac{1}{k \log^2 k} dk = \int_{\log 2}^{\infty} x^{-2} dx = [x^{-1}]_{\log 2}^{\infty} = \frac{1}{\log 2} < \infty$$

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Also

$$\sum_{k=3}^{\infty} \log^{-2q} k = \infty$$

by the following:

$$\int_3^\infty \log^{-2q} k dk = \int_{\log 3}^\infty y^{-2q} e^y dy = \infty \qquad \text{since } y^{-2q}e^y \to \infty.$$

Theorem 2.2. *Let G be a locally compact group. If G is not compact there exists a sequence of symmetric sets W_n with $e \in W_n$ such that $\lambda(W_n) \to \infty$.*

Proof. Let $U \subseteq G$ be compact with $e \in U$. There exists a compact symmetric set V such that $VV \subseteq U$ (see bachelorproject). Choose $g_1 \in G \setminus U$. Then $g_1 V \cap V = \emptyset$. Let $V_1 = g_1 V \cup V$ and $U_1 = g_1 U \cup U$. Next let $g_2 \in G \setminus U_1$. Again $g_2 V_1 \cap V_1 = \emptyset$. Defining $V_2 = g_1 V_1 \cup V_1$ and so on it's seen that

$$\lambda(V_n) = \sum_{k=0}^{n} \lambda(V) = (n+1)\lambda(V).$$

Since G is not compact it's always possible to find a new g_n. Therefore $\lambda(V_n) \to \infty$. Choosing $W_n = V_n \cup V_n^{-1}$ we get a sequence of compact symmetric subsets of G with $\lambda(W_n) \to \infty$. □

3. The L^p-conjecture.

Let me start by stating the theorem:

Theorem 3.1. *Let G be a locally compact Hausdorff group. If there exists a $p \in]1, \infty[$ such that $f * g \in L^p(G)$ for all symmetric functions $f, g \in L^p(G)$, then G is compact.*

Building up to the proof we need some lemmas, but first an introduction to the notation used. A function f is symmetric if $f(x) = f(x^{-1})$ for all $x \in G$. The Haar measure on G is λ and the measure of a measurable set A is $\lambda(A)$. 1_A is the characteristic function for any subset A of G.

Lemma 3.2. *For a compact symmetric subset A of G and $m, n \geq 1$, we have*

$$\lambda(A)^2 \lambda(A^{m+n}) \leq \lambda(A^4)\lambda(A^m)\lambda(A^n). \tag{2}$$

Proof. Let $m \geq 1$. For $k, l \geq 0$, $k \leq m$ and $x \in A^{l+2k}$. (this means that $x = abc$ with $a, b \in A^k$ and $c \in A^l$, with $A^0 = \{e\}$), follows :

$$\begin{aligned}
1_{A^m} * 1_{A^{m+l}}(x) &= \int 1_{A^m}(y) * 1_{A^{m+l}}(x-y)dy \\
&= \lambda(A^m \cap xA^{m+l}) \qquad \text{since } A = A^{-1} \\
&= \lambda(A^m \cap abcA^{m+l}) \\
&\geq \lambda(A^m \cap abA^m) \\
&= \lambda(a^{-1}A^m \cap bA^m) \qquad \lambda \text{ is a Haar-measure} \\
&\geq \lambda(A^{m-k})
\end{aligned}$$

It was twice used, that $aA^m \supseteq A^{m-k}$ for all $a \in A^k$. This is true because $A^m = \cup_{a \in A^k} aA^{m-k}$, which means that $a^{-1}A^m \supseteq A^{m-k}$ for all $a \in A^k$ and at last note that $A = A^{-1}$. Integrating over A^{l+2k} gives

$$\begin{aligned}
\lambda(A^{m-k})\lambda(A^{l+2k}) &\leq \int_{A^{l+2k}} 1_{A^m} * 1_{A^{m+l}}(x)dx \\
&\leq \int_G 1_{A^m} * 1_{A^{m+l}}(x)dx = \lambda(A^m)\lambda(A^{m+l}).
\end{aligned}$$

With $k = m - 1$ this is:

$$\lambda(A)\lambda(A^{l+2m-2}) \leq \lambda(A^m)\lambda(A^{m+l}). \tag{3}$$

Inserting $m = 4$ into (3) gives $\lambda(A)\lambda(A^{l+6}) \leq \lambda(A^4)\lambda(A^{l+4})$. Since $l \geq 1$ substitution with $j \geq 6$ gives

$$\lambda(A)\lambda(A^j) \leq \lambda(A^4)\lambda(A^{j-2}). \tag{4}$$

The inequality obviously holds for $j = 3$ adn $j = 4$. For $j = 5$ look at (3) with $m = 3$ and $l = 1$. Since the inequality in question clearly holds for $m = n = 1$, we can now look at $m + n \geq 3$ with $m \leq n$:

$$\begin{aligned}
\lambda(A)^2\lambda(A^{m+n}) &\leq \lambda(A)\lambda(A^4)\lambda(A^{m+n-2}) \qquad \text{by (4)} \\
&= \lambda(A)\lambda(A^4)\lambda(A^{2m+l-2}) \qquad l = n - m \\
&\leq \lambda(A^4)\lambda(A^m)\lambda(A^n) \qquad \text{by (3).}
\end{aligned}$$

□

From now on we use $p' = p/(p-1)$. With $p' = 1$ if $p = \infty$.

Lemma 3.3. *Let $p, q, r \in [1, \infty]$ be such that $p^{-1} + q^{-1} + r^{-1} \neq 1$. If $L^p_s * L^q_s \subseteq L_r$ ($f * g \in L_r$ for all $f \in L_p, g \in L_q$), then G is unimodular, $L^p * L^q \subseteq L_r$ and there is a $C_0 > 0$ such that*

$$\|f * g\|_r \leq C_0\|f\|_p\|g\|_q \qquad \text{for } f \in L^p \text{ and } g \in L^q. \tag{5}$$

Proof. Let $f \in L^p_s$ and $g \in L^q_s$. Since $T_f : g \mapsto f * g$ is linear and $\|T_f\| = \sup\{\|f * g\|_r \|g\|_q \leq 1\} < \infty$, there is a C_f such that $\|T_f g\| =$

$\|f * g\|_r \leq C_f\|g\|_q$. If $\|f\|_p = 0$ then $\|T_f\| = 0$. By inserting $C = C_f/\|f\|_p$ the following holds

$$\|f * g\|_r \leq C\|f\|_p\|g\|_q, \qquad \text{with } C < \infty. \tag{6}$$

for all symmetric f, g.

First notice that for $f \in L^r$, the calcule applies

$$\begin{aligned} f * \delta_b(x) &= \int f(y)\delta_b(y^{-1}x)dy \\ &= \int \Delta(y^{-1})f(y^{-1})\delta_b(yx)dy \\ &= \int \Delta(y^{-1}x)f(xy^{-1})\delta_b(y)\Delta(x^{-1})dy \\ &= \int \Delta(y^{-1})f(xy^{-1})\delta_b(y)dy \\ &= f(xb^{-1})\Delta(b^{-1}) \end{aligned}$$

and therefore

$$\begin{aligned} \|f * \delta_b\|_r^r &= \Delta(b^{-p}) \int \|f(xb^{-1})\|^r dx \\ &= \Delta(b^{-p})\Delta(b) \int \|f(x)\|^r dx = \Delta(b)^{1-p}\|f\|_r^r \end{aligned}$$

By similar calculations it is seen that

$$\begin{aligned} \Delta(a)^{1/r'}\|f * g\|_r &= \|f * g * \delta_b\|_r \\ &= \|(\delta_a * f * \delta_b) * (\delta_a * g * \delta_b)\|_r \\ &\leq C\|\delta_a * f * \delta_b\|_p\|\delta_a * g * \delta_b\|_q \\ &= C\Delta(a)^{1/p'}\|f\|_p\Delta(a)^{1/q'}\|g\|_q \end{aligned}$$

Since $f * g$ is a possitive function and $1/r' \neq 1/p' + 1/q'$ it follows that $\Delta(a) \geq \epsilon > 0$ for all $a \in G$. By Theorem 2.1 G is unimodular, and thus $\|f\|_p = \|\check{f}\|_p$ and $\|g\|_q = \|\check{g}\|_q$ for $f \in L^p$ and $g \in L^q$. Therefore

$$\begin{aligned} \|f * g\|_r &\leq \||f| * |g|\|_r \leq \|(|f| + |\check{f}|) * (|g| + |\check{g}|)\|_r \\ &\leq C\||f| + |\check{f}|\|_p\||g| + |\check{g}|\|_q \leq 4C\|f\|_p\|g\|_q. \end{aligned}$$

Take $C_0 = 4C$ and the lemma is proved. □

Lemma 3.4. *Let p, q, r and C_0 be as in Lemma 3.3 then for all compact $A, B \subseteq G$ the following holds*

$$(\lambda(A)\lambda(B))^{1/p'+1/q'} \leq C_0^2\lambda(AB)^{2/r'}. \tag{7}$$

Proof.

$$\begin{aligned}
\lambda(A)\lambda(B) &= \int 1_A * 1_B(x)dx \\
&= \int 1_{AB}(x) 1_A * 1_B(x)dx \qquad \text{since } 1_A * 1_B = 0 \text{ off } AB \\
&\leq \lambda(AB)^{1/r'} \|1_A * 1_B\|_r \qquad \text{Hölder's inequality} \\
&\leq C_0 \lambda(AB)^{1/r'} \|1_A\|_p \|1_B\|_q \qquad \text{Lemma3.3} \\
&= C_0 \lambda(AB)^{1/r'} \lambda(A)^{1/p} \lambda(B)^{1/q}
\end{aligned}$$

This gives

$$\lambda(A)^{1/p'} \lambda(B)^{1/q'} \leq C_0 \lambda(AB)^{1/r'}. \tag{8}$$

Since G is unimodular the following holds for $f \in L^q$ and $g \in L^p$

$$\begin{aligned}
\|f * g\|_r = \|(f * g)\check{}\|_r &= \|\check{f} * \check{g}\|_r \\
&\leq C_0 \|\check{g}\|_p \|\check{f}\|_q = C_0 \|f\|_q \|g\|_p
\end{aligned}$$

That gives

$$\lambda(A)^{1/q'} \lambda(B)^{1/p'} \leq C_0 \lambda(AB)^{1/r'}. \tag{9}$$

By multiplying (8) and (9) the proof is complete. □

Proof of the L_p-conjecture. Assume that $1 < p < \infty$ and $L^p_s * L^p_s \subseteq L^p$. This is a proof by contradiction, so let us assume that G is not compact. By lemma 3.3 with $p = q = r$ the group G is unimodular, $L^p * L^p \subseteq L^p$ and there exists a $C_0 > 0$:

$$\|f * g\|_p \leq C_0 \|f\|_p \|g\|_p \qquad \text{for } f, g \in L^p.$$

Lemma 3.4 tells us, that for all compact $A, B \subseteq G$,

$$\lambda(A)\lambda(B) \leq C_0^{p'} \lambda(AB).$$

From now on let $q = p'$. Since G is not compact there exists a compact and symmetric $A \subseteq G$ with $e \in A$, with

$$\lambda(A) > 1 \qquad \text{and} \qquad C_0^q / \lambda(A) < 2^{-(p+q)}.$$

For each $n \geq 2$ define

$$\begin{aligned}
a_n &= (nlog^2 n \lambda(A^n))^{-1/p}, \\
b_n &= (nlog^2 n \lambda(A^n))^{-1/q}, \\
f(x) &= \sum_{n=2}^{\infty} a_n 1_{A^n}(x), \\
g(x) &= \sum_{n=2}^{\infty} b_n 1_{A^n}(x).
\end{aligned}$$

I will now show, that $f \in L^p$:

$$\frac{a_{n+1}^p}{a_n^p} = \frac{nlog^2 n\lambda(A^n)}{(n+1)log^2(n+1)\lambda(A^{n+1})} \le \frac{\lambda(A^n)}{\lambda(A^{n+1})} \le C_0^q/\lambda(A) < 2^{-p}$$

This gives:

$$\sum_{n=k}^{\infty} a_n \le 2\sum_{n=k}^{\infty}(a_n - a_{n+1}) = 2a_k,$$

since $a_n - a_{n+1} = a_n(1 - a_{n+1}/a_n) \ge a_n/2$. I now want to rewrite f into a sum of orthogonal functions. f is in the vectorspace with basis $\{1_{A^n}\}_{n\ge 2}$. An orthogonal basis is easily found: $\{1_{A^n} - 1_{A^{n-1}}\}_{n\ge 3} \cap \{1_{A^2}\}$. Name it $\{u_n\}_{n\ge 2}$, an calculate the inner products with f:

$$(f, u_2) = \sum_{k=2}^{\infty} a_k \qquad \text{and} \qquad (f, u_n)_{n\ge 3} = \sum_{k=n}^{\infty} a_k,$$

and therefore

$$f = (\sum_{k=2}^{\infty} a_k)1_{A^2} + \sum_{n=3}^{\infty}(\sum_{k=2}^{\infty} a_k)(1_{A^n} - 1_{A^{n-1}}).$$

It then follows, that

$$\begin{aligned}
\|f\|_p^p &= (\sum_{k=2}^{\infty} a_k)^p \lambda(A^2) + \sum_{n=3}^{\infty}(\sum_{k=2}^{\infty} a_k)^p(\lambda(A^n) - \lambda(A^{n-1})) \\
&\le 2^p(a_2^p\lambda(A^2) + \sum_{n=3}^{\infty} a_n^p \lambda(A^n)) \\
&= 2^p \sum_{n=2}^{\infty}(n\log^2 n)^{-1} < \infty.
\end{aligned}$$

The same sort of calculations can be done for g and this shows, that $f \in L^p$ and $g \in L^q$. I will now show, that $f * g \in L^q$. Let $h \in L^p$, then

$$\begin{aligned}
|\int h(x)(f*g)(x)dx| &= |\int h(x)\int f(y)g(y^{-1}x)dydx| \\
&= |\int g(y^{-1})\int h(x)f(x^{-1}y)dxdy| \\
&= |\int (h*f)(y)g(y^{-1})dy| \\
&leq\|h*f\|_p\|g\|_q \\
\le C_0\|h\|_p\|f\|_p\|g\|_q
\end{aligned}$$

Theorem 6.16 in [3] then tells us that $f * g \in L^q$. The last part of the proof will show that $\|f * g\|_q = \infty$ therefore yielding the desired contradiction. For $m, k \ge 1$ and $x \in A^k$

$$1_{A^m} * 1_{A^{m+k}}(x) = \lambda(A^m \cap xA^{m+k}) \ge \lambda(A^m).$$

which gives the following

$$\begin{aligned}(f*g)(x) &= \sum_{m=2}^{\infty}\sum_{n=2}^{\infty} a_m b_n (1_{A^m} * 1_{A^n})(x) \\ &\geq \sum_{k=2}^{\infty}\sum_{m=2}^{\infty} a_m b_{m+k} (1_{A^m} * 1_{A^{m+k}})(x) \\ &\geq \sum_{k=2}^{\infty}\sum_{m=2}^{\infty} a_m b_{m+k} \lambda(A^m) 1_{A^k}(x).\end{aligned}$$

Therefore

$$\begin{aligned}\|f*g\|_q^q &\geq \sum_{k=2}^{\infty}(\sum_{m=2}^{\infty} a_m b_{m+k}\lambda(A^m))^q \int 1_{A^k}(x)dx \\ &= \sum_{k=2}^{\infty}(\sum_{m=2}^{\infty} a_m b_{m+k}\lambda(A^m))^q \lambda(A^K). \qquad (10)\end{aligned}$$

Since $\lambda(A) > 1$ lemma 3.2 gives

$$\lambda(A^{m+k}) \leq \lambda(A^4)\lambda(A^m)\lambda(A^k), \qquad \text{for } m, k \geq 1.$$

Pick only pairs (m, k) such that $3 \leq k \leq m \leq 2k$, then

$$(m+k)\log^2(m+k) \leq 3k\log^2(3k) \leq 3k(2log(k))^2 = 12k\log^2 k.$$

This means, that

$$b_{m+k} \geq (12\lambda(A^4)\lambda(A^m)\lambda(A^k)k\log^2 k)^{-1/q}$$

and by similar calculations

$$a_m \geq (8\lambda(A^m)k\log^2 k)^{-1/p} \geq (12\lambda(A^m)k\log^2 k)^{-1/p}$$

which helps us establish:

$$a_m b_{m+k} \geq (12\lambda(A^4)\lambda(A^m)\lambda(A^k)^{1/q}k\log^2 k)^{-1}$$

With $C^{-1} = 12\lambda(A^4)$ and combining with (10)

$$\begin{aligned}\|f*g\|_q^q &\geq \sum_{k=3}^{\infty}(\sum_{m=k}^{2k} \frac{C\lambda(A^m)}{\lambda(A^m)k\log^2 k\lambda(A^k)^{1/q}})^q\lambda(A^k) \\ &= C^q\sum_{k=3}^{\infty}(\log k)^{-2q} = \infty\end{aligned}$$

□

4. Young's inequality.

Theorem 4.1 (Young's inequality)**.** *Let G be a locally compact group. Let $p, q \in]1, \infty[$ such that $1/p+1/q > 1$ and define r by $1/p+1/q-1/r = 1$. for all $f \in L^p(G)$ and $g \in L^q(G)$ weh have*

(a) $f * g \in L^r(G)$
(b) $\|f * g\|_r \leq \|f\|_p \|g\|_q$.

Lemma 4.2. *If $f_1, \dots, f_n \in L^1$ and $a_1, \dots, a_n$ are positive, then $f_1^{a_1} \cdots f_n^{a_n} \in L^{(a_1+\cdots+a_n)^{-1}}$ and*

$$\|f_1^{a_1} \cdots f_n^{a_n}\|_{(a_1+\cdots+a_n)^{-1}} \leq \|f_1\|_1^{a_1} \cdots \|f_n\|_1^{a_n}.$$

Proof. This will be proved by induction. First notice, that if $f \in L^1$ and $x > 0$ then $f^x \in L^{x^{-1}}$ and $\|f^x\|_{x^{-1}} = \|f\|_1^x$, since $\int |f^x|^{x^{-1}} d\lambda = \|f\|_1$. Now assume that the claim holds for $n-1$.

$$\int |f_1^{a_1} \cdots f_n^{a_n}|^{(a_1+\cdots+a_n)^{-1}} d\lambda = \int |f_1^{b_1} \cdots f_{n-1}^{b_{n-1}}||f_n^{b_n}| d\lambda \tag{11}$$

where $b_i = a_i/(a_1 + \cdots + a_n)$. By induction $f_1^{b_1} \cdots f_{n-1}^{b_{n-1}} \in L^{(a_1+\cdots+a_n)^{-1}}$ and $f_n \in L^{b_n^{-1}}$. Since $b_1 + \cdots + b_n = 1$ Hölder's inequality gives

$$\begin{aligned} \int |f_1^{b_1} \cdots f_{n-1}^{b_{n-1}}||f_n^{b_n}| d\lambda &\leq \|f_1^{b_1} \cdots f_{n-1}^{b_{n-1}}\|_{b_1+\cdots+b_{n-1}}^{-1} \|f_n^{b_n}\|_{b_n} \\ &\leq \|f_1\|_1^{b_1} \cdots \|f_n\|_1^{b_n} \qquad \text{by induction} \\ &= (\|f_1\|_1^{a_1} \cdots \|f_n\|_1^{a_n})^{(a_1 \cdots a_n)^{-1}} \qquad (12) \end{aligned}$$

The expressions (11) and (12) show that

$$\|f_1^{a_1} \cdots f_n^{a_n}\|_{(a_1+\cdots+a_n)^{-1}} \leq \|f_1\|_1^{a_1} \cdots \|f_n\|_1^{a_n}$$

as desired. □

Proof of Young's inequality. Let $f \in L^p$ and $g \in L^q$. Let us first show, that $f * g$ exists and is finite λ-almost everywhere. Given an $x \in G$ we rewrite

$$\begin{aligned} &\int |f(xy)g(y^{-1})| dy \\ &\quad = \int (|f(xy)|^p |g(y^{-1})|^q)^{1/r} |f(xy)|^{1-p/r} |g(y^{-1})|^{1-q/r} dy \\ &\quad = \int (|f(xy)|^p |g(y^{-1})|^q)^{1/r} (|f(xy)|^p)^{1/p-1/r} (|g(y^{-1})|^q)^{1/q-1/r} dy. \end{aligned}$$

Using the previous lemma with $a_1 = 1/r$, $a_2 = 1/p - 1/r$ and $a_3 = 1/q - 1/r$ tells us that

$$\int |f(xy)g(y^{-1})| dy \leq (\int |f(xy)g(y^{-1})| dy)^{1/r} (\|f\|_p^p)^{1/p-1/r} (\|g\|_q^q)^{1/q-1/r}$$

which leads to

$$(13)\qquad (\int |f(xy)g(y^{-1})|dy)^r \leq \|f\|_p^{r-p}\|g\|_q^{r-q} \int |f(xy)|^p|g(y^{-1})|^q dy$$

Since f, g are measurable and the integral is a continous function (and therefore measurable) $f * g$ is also measurable. Let $F = |f|^p$ and $G = |g|^q$ then $F, G \in L^1$. Also let $f_n, g_n \in C_c$ be given such that $f_n \to F$ and $g_n \to G$. By (Stetkrs noter) $F * G \in L^1$ and therefore $f_n * g_n \to F * G$. By Fubini's theorem

$$\begin{aligned}\|f_n * g_n\|_1 &= \int\int |f_n(xy)g_n(y^{-1})|dydx \\ &= \int \Delta(y^{-1})\|f_n\|_1|g(y^{-1})|dy \\ &= \|f_n\|_1\|g_n\|_1.\end{aligned}$$

Since $\|F * G\|_1$ exists $\|f_n * g_n\|_1 \to \|F * G\|_1$. Also $\|f_n\|_1 \to \|F\|_1$ and $\|g_n\|_1 \to \|G\|_1$ which shows that $\|F * G\|_1 = \|F\|_1\|G\|_1 = \|f\|_p^p\|g\|_q^q$. Thus we can integrate bot sides of (13) obtaining

$$\begin{aligned}\int |f * g(x)|^r dx &\leq \int (\int |f(xy)g(y^{-1})|dy)^r dx \\ &\leq \|f\|_p^{r-p}\|g\|_q^{r-q}\|f\|_p^p\|g\|_q^q \\ &= \|f\|_p^r\|g\|_q^r.\end{aligned}$$

This is Young's inequality. □

5. Sharpness of Young's inequality.

An interesting thing is to investigate for which r Young's inequality holds. I will prove the following theorem:

Theorem 5.1. *Let $p, q, r \in [1, \infty]$ and $p > 1$. Suppose that G is an infinite locally compact abelian group and that $L^p * L^q \subseteq L^r$.*

(a) *If G is discrete then $1/r \leq 1/p + 1/q - 1$.*
(b) *If G is compact then $1/r \geq 1/p + 1/q - 1$.*
(c) *If G is neither discrete nor compact then $1/r = 1/p + 1/q - 1$.*

To prove this we need some lemmas. Also define

$$\|f\|_{pu} = \max\{\|f\|_p, \|f\|_u\}, \qquad \text{for } f \in L^p \cap C(G) \text{ with } p > 1.$$

The article by Saeki [1] uses C_0 in place of C, but that must be the same.

Lemma 5.2. *Let G be a locally compact group. Suppose that $p, q, r \in [1, \infty]$ and $p > 1$. If $(L^p \cap C_0) * (L^q \cap C_0) \subseteq L^r$, then G is unimodular and there exists a finite positive constant C_1 such that*

$$\|f * g\|_r \leq C_1\|f\|_{pu}\|g\|_{pu} \qquad \textit{for } f \in L^p \cap C_0 \textit{ and } g \in L^q \cap C_0.$$

If G is also noncompact, then $r \geq \max\{p, q\}$.

Proof. The constant C_1 is found like in beginning og the proof of Lemma 3.3. Pick nonzero and positive $f, g \in C_c(G)$ and any $a \in G$. With $b = a^{-1}$ the following holds:

$$\begin{aligned}\|f * g\|_r &= \|f * \delta_b * \delta_a * g\|_r \\ &\leq C_1 \|f * \delta_b\|_{pu} \|\delta_a * g\|_{qu} \\ &= C_1 \max\{\Delta(a)^{1/p'} \|f\|_p, \Delta(a)\|f\|_u\} \|g\|_{qu}.\end{aligned}$$

Since $p' < \infty$ G must be unimodular.

Now assume that G is noncompact. With f, g as before $f * g$ is in $C_c(G)$. Since G is noncompact we can find $a_1, \dots, a_n \in G$ such that all the functions $\delta_{a_k} * f$ and $\delta_{a_k} * f * g$ have disjoint supports. Then

$$\begin{aligned}n^{1/r} \|f * g\|_r &= \left\| \sum_{k=1}^{n} \delta_{a_k} * f * g \right\|_r \\ &\leq C_1 \left\| \sum_{k=1}^{n} \delta_{a_k} * f \right\|_{pu} \|g\|_{qu} \\ &= C_1 \max\{n^{1/p} \|f\|_p, \|f\|_u\} \|g\|_{qu}\end{aligned}$$

for any $n \geq 1$. From a certain n the part $n^{1/p}\|f\|_p$ will dominate. Since f, g are nonzero functions $\|f * g\|_r > 0$, and the inequality holds for any n it follows that $r \geq p$. G is unimodular so p, q can be interchanged (see the proof of Lemma 3.4) similarily giving $r \geq q$. □

Lemma 5.3. *Let G, p, q, r and C_1 be as in Lemma 5.2. Then*

$$(\lambda(A)\lambda(B))^{1/p'+1/q'} \leq C_1^2 \lambda(AB)^{2/r'}$$

for all compact subsets A, B of G with $\lambda(A), \lambda(B) \geq 1$.

Proof. The proof is very similar to the proof of Lemma 3.4. □

Lemma 5.4. *Let G be a noncompact group and $p \in]1, \infty[$. If for each given $\epsilon > 0$ exists a compact subset A of G , with $\lambda(A)$ sufficiently large, such that*

$$\liminf_{n \to \infty} (n^{-1} \log \log \lambda(A^{2^n})) < \epsilon, \tag{14}$$

for all $r, q \in [1, \infty]$ satisfying

$$\frac{1}{r} > \frac{1}{p} + \frac{1}{q} - 1.$$

there exists $f \in L^p_s \cap C_0^+(G)$ such that

$$f * L^q_s \not\subseteq L^r$$

Remark 5.5. *To see what "sufficiently large" means you have to look at the proof. It involves a constant C_3 and the requirement is, that $\log \log C_3 \lambda(A)$ is defined. The reason this can't be included in the lemma is that it is proved by contradiction. Since we will use the lemma* *on discrete groups (the measure is counting) it's enough for us to show, that we can always find a bigger A such that (14) holds. Then we won't have to worry about the constant C_3.*

Remark 5.6. *Also the lemma is a bit weaker than the Theorem 2 in* [1]*, since this is all that's necessary to prove Theorem 5.1. In the article* [1] *is shown that it's possible to find a single f for all p, q, r for which*

$$\frac{1}{r} > \frac{1}{p} + \frac{1}{q} - 1.$$

Proof of Lemma 5.4. This is proved by contradiction. Let p, q, r be given such that $1/r > 1/p + 1/q$. Now assume that

$$(L^p_s \cap C_0) * (L^q_s \cap C_0) \subseteq L^r.$$

Like the proof of Lemma 3.3 this means that

$$(L^p \cap C_0) * (L^q \cap C_0) \subseteq L^r.$$

By the two previous lemmas there is a $C_1 > 0$ such that for all compact $A, B \subseteq G$ with $lambda(A), \lambda(B) > 1$ the following holds

$$(\lambda(A)\lambda(B))^{1/p'+1/q'} \leq C_1^2 \lambda(AB)^{2/r'}$$

Since $p' < \infty$ and G is noncompact, the left hand side can be chosen bigger than C_1^2, but then $r' < \infty$. Hence we can define

$$\beta = r'(1/p' + 1/q')$$

and notice that the restrictions on p, q, r is equivalent with $\beta > 1$. Let $C_2 = C_1^{r'}$ and $A = B$ then $\lambda(A)^\beta \leq C_2 \lambda(A^2)$ for all compact A with $\lambda(A) > 1$. Recursive use of this inequality gives

$$\lambda(A)^{\beta^n} \leq C_2^{\beta^{n-1}+\beta^{n-2}+\cdots+1} \lambda(A^{2^n}) = C_2^{(\beta^n-1)/(\beta-1)} \lambda(A^{2^n})$$

letting $C_3 = C_2^{1/(1-\beta)}$ this is equivalent with

$$(C_3\lambda(A))^{\beta^n} \leq C_3 \lambda(A^{2^n}).$$

As mentioned in Remark 5.5 let us assume that $\lambda(A)$ is sufficiently large, then we get

$$\log\log(C_3\lambda(A^{2^n}) \geq n \log\beta + \log\log(C_3\lambda(A)).$$

But that contradicts the requirement in the theorem. Thereby showing that

$$(L^p_s \cap C_0) * (L^q_s \cap C_0) \not\subseteq L^r$$

for all $q, r \geq 1$ with $1/r > 1/p + 1/q - 1$, and thus completing the proof. □

Lemma 5.7. *Let G_0 be a subgroup of G. If $L_p(G) * L_q(G) \subseteq L_r(G)$ then $L_p(G/G_0) * L_q(G/G_0) \subseteq L_r(G/G_0)$ with respect to the induced Haar meassure on G/G_0.*

Proof. Let κ be the canonical homomorphism $\kappa : G \to G/G_0 = A$. Define the measure λ_A by:

$$\int f(a)d\lambda_A(a) = \int f \circ \kappa(b)d\lambda_G(b).$$

This meassure is both positive and linear. It's also left-invariant, since

$$\begin{aligned}\int f \circ L_x(a)d\lambda_A(a) &= \int f \circ L_x \circ \kappa(b)d\lambda_G(b)\\ &= \int f \circ \kappa(x'b)d\lambda_G(b) \qquad \kappa(x') = x \text{ since } \kappa \text{ is onto}\\ &= \int f \circ \kappa(b)d\lambda_G(b) \qquad \lambda_G \text{ is Haar}\\ &= \int f \circ L_x(a)d\lambda_A(a).\end{aligned}$$

What we show for this meassure is therefore equivalent to showing it for the normalised Haar-meassure on G/G_0.

Let $f \in L_p(A)$ and $g \in L_q(A)$. Define $\tilde{f} = f \circ \kappa$ and $\tilde{g} = g \circ \kappa$. Since

$$\int |f \circ \kappa(x)|^p d\lambda_G(x) = \int |f(y)|^p d\lambda_A(y)$$

by definition, $\tilde{f} \in L_p(G)$ and $\tilde{g} \in L_q(G)$. Also if $f * g \circ \kappa \in L_r(G)$ then $f * g \in L_r(A)$. But that follows from:

$$\begin{aligned}L_r(G) \ni \tilde{f} * \tilde{g}(x) &= \int f(\kappa(y^{-1}x))g(\kappa(y))d\lambda_G(y)\\ &= \int f(\kappa(y)^{-1}\kappa(x))g(\kappa(y))d\lambda_G(y)\\ &= \int f(a^{-1}\kappa(x))g(a)d\lambda_A(a)\\ &= f * g \circ \kappa(x).\end{aligned}$$

□

Proof of Theorem 5.1. (a) Assume G is a torsion group (all elements have finite order). Then it's no problem finding A "big enough" and A^{2^n} has a limited size (from a certain n). Therefore Lemma 5.4 can be used for $p < \infty$. If G is not a torsion group, then there is a $g \in G$ such that $< g > \cong \mathbb{Z}$, where $< g >$ is the cyclic subgroup of G created by g. Choose $A = [0, m] \cap \mathbb{Z}$, then $A^{2^n} = [0, 2^n m] \cap \mathbb{Z}$, and $\lambda(A^{2^n}) = 2^n m + 1$. This A satisfies Lemma 5.4 with $p < \infty$. If $p = \infty$ then $q = 1$ and $r = \infty$ follows by inserting $f = 1_G \in L^\infty(G)$ in $f * g$.

(b) If G is compact, then G is eihter totally disconnected (if the dual of G is a torsion group, Theorem 2.5.6 [4]) or contains a compact subgroup G_0 such that $G/G_0 \cong \mathbb{T}$. The last follows by setting $G_0 = \ker \phi$ where ϕ is a character for which $\phi(G)$ is infinite (one such exists according to (24.26) in [2]). Since $G/G_0 \cong \phi(G)$ is a compact subset of $\mathbb{T}$, G/G_0 must be $\mathbb{T}$. In the following we can assume $r > 1$ for if $r = 1$ then it will always be true that $1/r \geq 1/p + 1/q + 1$ since $p, q \geq 1$. We can also assume that $1/r \neq 1/p+1/q+1$, cause otherwise it would simply be a special case of what we want to show. Let C_0 be the constant from Lemma 3.3. For all compact A Lemma 3.4 yields

$$\lambda(A)^{1/p'+1/q'} \leq C_0\lambda(A^2)^{1/r'}.$$

Since G is not discrete (and infinite) and $r' < \infty$ the right hand side can be smaller than 1, which tells us that $1/p' + 1/q' > 0$. Defining $\beta = r'(1/p' + 1/q')$ and $C_2 = C_0^{r'}$ this can be written as

$$\lambda(A)^{\beta} \leq C_2\lambda(A^2). \tag{15}$$

If G is totally disconnected, then every neghborhood of e contains a compact-open subgroup A (Theorem 2.4.4 in [4]). Then A can be chosen arbitrarily small and (15) is therefore only possible when $\beta \geq 1$.

If G is not totally disconnected it contains a compact subgroup G_0 such that $G/G_0 \cong \mathbb{T}$. Lemma 5.7 then tells us, that $L_p * L_q \subseteq L_r$ on $\mathbb{T}$. Using $A = [0, t]$ in (15) then gives $t^{\beta} \leq 2C_2t$ for all $t \in [0, \pi]$ (the article [1] says $[0, 2\pi]$, but the torus "wraps" at 2π). Again this is only possible when $\beta \geq 1$. In both cases it therefore follows that $1/r \geq 1/p + 1/q$.

(c) G is compact and infinite. As before we can suppose that $p < \infty$, since otherwise $q = 1$ and $r = \infty$. Consider the case when G contains an open subgroup of the form $\mathbb{R} \times H$ where H is a locally compact group. Then $L_p(\mathbb{R}) * L_q(\mathbb{R}) \subseteq L_r(\mathbb{R})$, since it holds for G. As in (b)

$$\lambda(A)^{1/p'+1/q'} \leq C_0\lambda(A^2)^{1/r'}, \qquad \text{for all compact } A \subseteq \mathbb{R}$$

Using $A = [0, t]$ for all $t \in \mathbb{R}$ tells us that $1/p + 1/q - 1 = 1/r$.

Assume G has no subgroup of the above form. Letting $G' = \cup_{n=1}^{\infty}(U \cup U^{-1})^n$ where U is a compact open neighbourhood of e, makes G' an open compactly generated subgroup of G. By (9.8) in [2] G' is topologically isomorphic with $\mathbb{Z}^a \times F$ for some nonnegative integer a and some compact abelian group F. Since $\{0\}$ is open in $\mathbb{Z}$ and F is open in F, $F = 0 \times F$ is open in $\mathbb{Z}^a \times F = G'$. This means that F is also open in G. G is nondiscrete so (b) applied to F gives $1/p + 1/q - 1 \leq 1/r$. Since F is open in G, (5.21) in [2] tells us that G/F is discrete. Also G is noncompact so G/F must be infinite (if G/F is finite, G can be expressed as finite union of xF where $x \in G/F$ contradicting the fact that G is noncompact). Thus (a) gives $1/p + 1/q - 1 \geq 1/r$.

This means that $1/p + 1/q - 1 = 1/r$ if G is neither discrete nor compact. □

References

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