Wavelet transforms and Besov Spaces related to symmetric cones

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Abstract

We present a general framework for construction of Banach spaces. This work is a generalization of the coorbit theory by Feichtinger and Gröchenig, and it unifies the coorbit theory with the theory for band-limited functions. The framework is then used to describe Besov spaces related to the forward light cone.

1. Introduction

In the 1980's Feichtinger and Gröchening [3, 4, 5] initiated the so-called coorbit-theory thus unifying the description of well-known Banach function spaces such as Besov spaces, Triebel-Lizorkin spaces and Bergman spaces. Not all Bergman spaces can be described using this theory, and the reason it fails is because of the non-integrability of a kernel function. Here we suggest an alternative coorbit-theory, which is able to account for a larger scale of Bergman spaces. This theory also includes the theory of band-limited functions, for which the kernel is also non-integrable. We present a wavelet description of Besov spaces related to the forward light cone in $\mathbb{R}^2$.

2. Wavelets on the cone

For simplicity let $\Lambda \subseteq \mathbb{R}^2$ be the forward light cone

$$\Lambda = \{(x,y) \in \mathbb{R}^2 \,|\, y^2 - x^2 > 0, y > 0\}$$

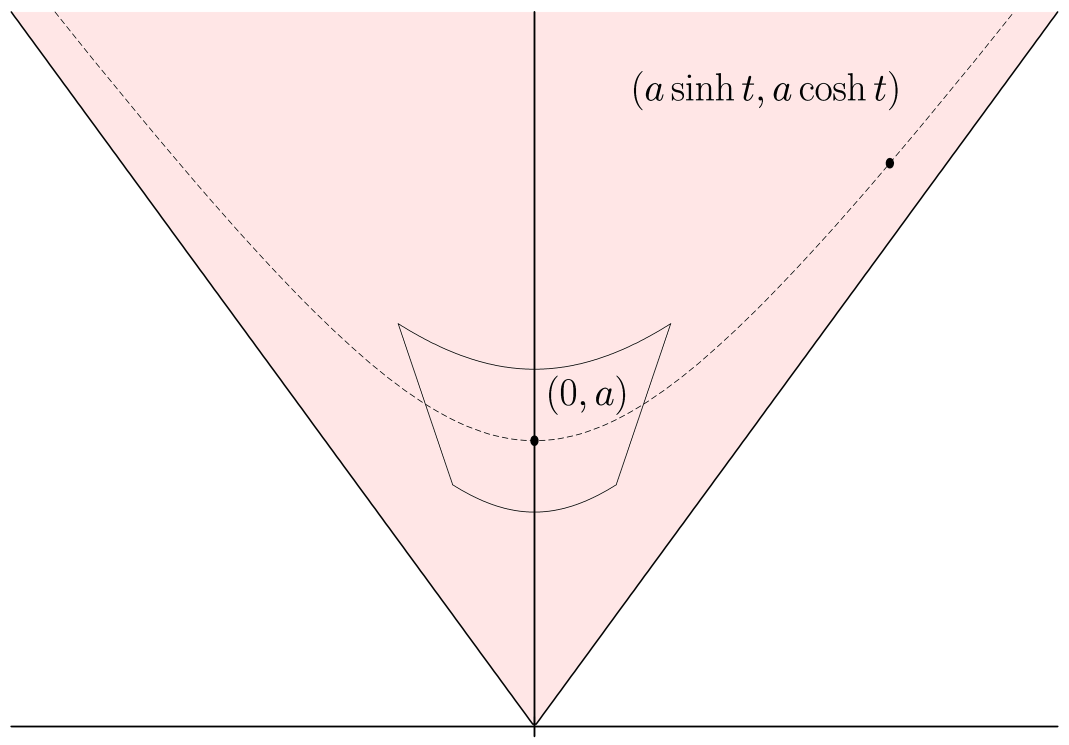

Figure 1: *The forward light cone in* $\mathbb{R}^2$

The group

$$H = \mathbb{R}_+ \mathrm{SO}(1,1) = \left\{ A = \begin{pmatrix} a\cosh t & a\sinh t \\ a\sinh t & a\cosh t \end{pmatrix} \,\middle|\, a \in \mathbb{R}_+, t \in \mathbb{R} \right\}$$

acts transitively on Λ. Therefore we can define a representation of

$$G = H \rtimes \mathbb{R}^2$$

on $L^2_\Lambda = \{f \in L^2(\mathbb{R}^2) \,|\, \mathrm{supp}(\widehat{f}) \subseteq \Lambda\}$ by

$$\pi(A,b)f(x) = \frac{1}{\sqrt{\det A}} f(A^{-1}(x-b))$$

Let $\psi \in \mathcal{S}_\Lambda = \mathcal{S}(\mathbb{R}^2) \cap L^2_\Lambda$ be such that

$$\mathrm{supp}(\widehat{\psi}) \subseteq \{A(a,t) \,|\, (a,t) \in [1/2, 3/2] \times [-1,1]\}$$

then the wavelet coefficients

$$W_\psi(f)(a,b) = \int_{\mathbb{R}^2} f(x) \overline{\pi(A,b)\psi(x)}\, dx$$

satisfy

$$|W_\psi(\psi)(a,t,b)| \leq \frac{Ca^j}{(1+a)^j (1+\cosh^2 t + \sinh^2 t)^m (1+|b|^2)^n} \qquad (1)$$

so $W_\psi(\psi) \in L^1\big(\frac{da\,dt\,db}{a^3}\big)$ and therefore we can normalize ψ such that

$$W_\psi(f) * W_\psi(\psi) = W_\psi(f)$$

and

$$L^2_\Lambda \simeq L^2(G) * W_\psi(\psi)$$

3. Banach Spaces

3.1 Construction

Let S be a Fréchet space continuously included in its conjugate dual S^* with dual pairing $\langle \cdot, \cdot \rangle$. Let (π, S) be a representation of the group G and Y a left-invariant Banach function space on G. Define

$$W_\psi(\Phi)(x) = \langle \Phi, \pi(x)\psi \rangle$$

Assume that $\psi \in S$ is cyclic and

R1. $W_\psi(\phi) * W_\psi(\psi) = W_\psi(\phi)$

R2. $Y \ni F \mapsto F * W_\psi(\psi) \in Y$ is continuous

R3. if $F = F * W_\psi(\psi) \in Y$ then $\pi(F)\psi \in S^*$.

R4. $\pi(\overline{W_\psi(\psi)})\psi \in S$

then we can define the space

$$\mathrm{Co}_S^\psi Y = \{\Phi \in S^* \,|\, W_\psi(\Phi) \in Y\}$$

Main Theorem. *1.* $\mathrm{Co}_S^\psi Y$ *is a* π*-invariant Banach space*

2. $W_\psi : \mathrm{Co}_S^\psi Y \to Y * W_\psi(\psi)$ *is an isometric isomorphism*

3. $\mathrm{Co}_S^\psi Y$ *depends very little on* ψ *and* S

3.2 Discretization

Given a nbh. U and points x_i assume that $\sum_i \psi_i = 1$ and $\mathrm{supp}(\psi_i) \subseteq x_i U$ (and that there is only finite overlap between $x_i U$). We investigate if the operator $T : Y * W_\psi(\psi) \to Y * W_\psi(\psi)$

$$TF = \sum_i F(x_i)\psi_i * W_\psi(\psi)$$

is close to I. If so it is invertible and

$$T^{-1} = \sum_{k=0}^{\infty} (I-T)^k$$

This enables us to reconstruct $\Phi \in \mathrm{Co}_S^\psi Y$ from samples of $W_\psi(\Phi)$.

Remark. *Feichtinger and Gröchenig showed that for integrable representations, we can always pick* U *small enough and matching* $\{x_i\}$ *such that* T *is invertible. However this can also be proven for the non-integrable discrete series representations of* $SU(1,1)$ *as in [2] in the construction of Bergman spaces.*

4. Besov Spaces

4.1 Definition

There are points $\{c_j\}$ and balls $B_r(c_j)$ of fixed radius r such that $B_{r/2}(c_j)$ do not overlap, yet $B_r(c_j)$ cover Λ. Let ϕ_j be functions such that $\mathrm{supp}(\widehat{\phi}_j) \subseteq B(c_j)$ and $\sum_j \widehat{\phi}_j = 1$.

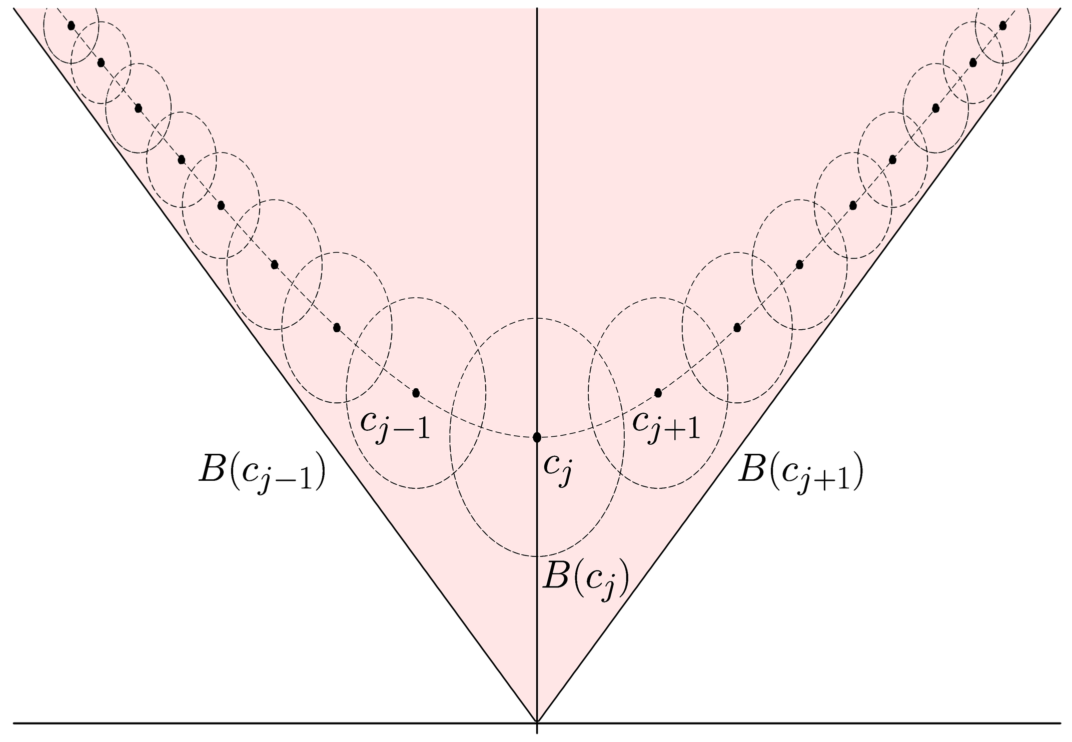

Figure 2: *Whitney cover of light cone in* $\mathbb{R}^2$

Then define the Besov spaces on the cone as in [1] to be

$$B_s^{p,q} = \left\{ f \in \mathcal{S}'_\Lambda / \mathcal{S}'_{\partial\Lambda} \,\middle|\, \|f\|_{B_s^{p,q}} = \Big(\sum_j \Delta(c_j)^{-s} \|f * \psi_j\|_p^q \Big)^{1/q} < \infty \right\}$$

4.2 As Coorbits

The estimate (1) ensures that for the space

$$L_s^{p,q} = \left\{ f \,\middle|\, \Big(\int_{\mathbb{R}_+} \int_{\mathbb{R}} \Big(\int_{\mathbb{R}^2} |f(a,t,b)|^p\, db \Big)^{q/p} a^{sq} \frac{dt\, da}{a^3} \Big)^{1/q} < \infty \right\}$$

the conditions (R2) and (R3) are satisfied. Further it ensures that (R1) and (R4) are satisfied for $(S_\Lambda)^* \simeq (S^*)_{\bar{\Lambda}} / S_{\partial\Lambda}$. Therefore we can define

$$\mathrm{Co}_S^\psi L_s^{p,q} = \{\Phi \in (S_\Lambda)^* \,|\, W_\psi(\Phi) \in L_s^{p,q}\}$$

It can then be shown that

$$B_s^{p,q} = \mathrm{Co}_S^\psi L_{s'}^{p,q}$$

with equivalent norms when $s' = s + (1+3q)/2$.

5. Further research

- **Bergman spaces** We aim to describe the Bergman spaces on $\mathbb{R}^2 + i\Lambda$ as coorbits and generalize to symmetric cones. Here the removal of integrability from the coorbit theory will be important.
- **Quasi Banach spaces** Show that the above theory can describe quasi Banach versions of both modulation, Besov and Bergman spaces.
- **Homogeneous space** There is already a coorbit theory on homogeneous space. We aim to remove integrability from it and treat nilpotent groups as special cases.

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