Semilinear evolution equations on resistance spaces

Joe P. Chen

Colgate UNIVERSITY

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Joint work with

Michael Hinz
Bielefeld

Alexander Teplyaev
UConn

Hydrodynamic program on singular spaces

Goal: Rigorous derivation of fluid equations from interacting particle systems on singular spaces, such as fractals.
Microscopic model: the exclusion process (an ergodic Markov chain) on an infinite weighted graph
Macroscopic PDE: a nonlinear heat equation in the diffusive scaling limit

The entire program is presented in 4 parts.

1. The moving particle lemma for the exclusion process on a finite weighted graph
   (the analog of Thomson's inequality for random walks—a Sobolev embedding theorem)
   C. '16, `arXiv:1606.01577`. *Electron. Commun. Probab.* **22** (2017) paper no. 47.
2. Local ergodic theorem for the exclusion process on strongly recurrent graphs
   C. '17, `arXiv:1705.10290`.
3. **Semilinear evolution equations on resistance spaces** (this talk)
   C.–Hinz–Teplyaev '17+
4. Hydrodynamic limit (LLN, LDP) of the exclusion process on the Sierpinski gasket
   C.–Hinz–Teplyaev '17+

These results are summarized in the review
C.–Hinz–Teplyaev '17, `arXiv:1702.03376`. To appear in the proceedings for the conference "SPDEs and Related Fields" in honor of Michael Röckner's 60th birthday (2017+)

Parabolic evolution problems

Let $\sigma > 0$, V be a vector field on $\mathbb{R}^n$, and $\Phi : \mathbb{R} \to \mathbb{R}$ be a (nonlinear) Lipschitz function.

$$\partial_t w = \sigma \Delta w + \mathrm{div}(\Phi(w) V)$$

Under suitable conditions on Φ and V, as well as the initial conditions, we establish existence, uniqueness, and regularity of solutions.

- **Classical case:** One can use the chain rule

  $$\mathrm{div}(\Phi(w)V) = \Phi'(w)\langle V, \nabla w\rangle + \Phi(w)\mathrm{div} V,$$

  then exploit the smoothness of w, V, and Φ to obtain estimates, *cf.* [Evans PDE, Ch. 7].
- **Singular case:** Carré du champ does not exist, *i.e.,* the Dirichlet energy

  $$\mathcal{E}(u,u) = \int d\Gamma(u,u) \;\text{ cannot be literally written as }\; \int |\nabla u|^2\, dx$$

  Energy measure $d\Gamma(u,u)$ is singular w.r.t. the reference measure dx.
  No pointwise formula for ∇u. Expressions such as $|\nabla u|^2\, dx$ or $\langle V, \nabla v\rangle dx$ are to be understood as measures. What about divergence?

  A fix: Use the theory of Dirichlet forms. We will concentrate on low-dimensional settings, in which case Kigami's **resistance forms** apply.

Treating the singular case: Resistance form & Sobolev embedding

[Kigami, early 2000's]

Let K be a nonempty set. A **resistance form** $(\mathcal{E}, \mathcal{F})$ on K is a pair such that

1. $\mathcal{F}$ is a vector space of $\mathbb{R}$-valued functions on K containing the constants, and $\mathcal{E}$ is a nonnegative definite symmetric quadratic form on $\mathcal{F}$ satisfying
$$\mathcal{E}(u,u) = 0 \iff u \text{ is constant.}$$
2. $\mathcal{F}/\{\text{constants}\}$ is a Hilbert space with norm $\mathcal{E}(u,u)^{1/2}$.
3. Given a finite subset $V \subset K$ and a function $v : V \to \mathbb{R}$, there is $u \in \mathcal{F}$ s.t. $u|_V = v$.
4. For $x, y \in K$,
$$R(x,y) := \sup \left\{ \frac{[u(x)-u(y)]^2}{\mathcal{E}(u,u)} : u \in \mathcal{F},\ \mathcal{E}(u,u) > 0 \right\} < \infty.$$
5. If $u \in \mathcal{F}$, then $\bar{u} := 0 \vee (u \wedge 1) \in \mathcal{F}$ and $\mathcal{E}(\bar{u},\bar{u}) \leq \mathcal{E}(u,u)$.

(K, R) is a metric space. Can always assumed to be complete.

Note that Item 4 implies that $|u(x) - u(y)| \leq R(x,y)^{1/2}\mathcal{E}(u,u)^{1/2}$, which then implies the Sobolev embedding $\mathcal{F} \subset C(K)$.

The classical Dirichlet form in $\mathbb{R}^n$ is a resistance form iff $n = 1$.

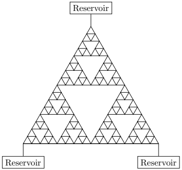
Resistance space = space K equipped with a resistance form $(\mathcal{E}, \mathcal{F})$

Standing Assumption. (K, R) is compact and connected; μ is a finite Borel measure on K.

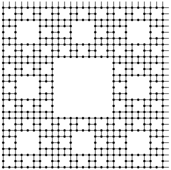
Then

- Every function in $C(K)$ is bounded; thus $\mathcal{F}$ is an algebra under pointwise multiplication.
- $(\mathcal{E}, \mathcal{F})$ is a regular Dirichlet form on $L^2(K, \mu)$. [Think: $\mathcal{E}(u, v) =$ "$\int_K \nabla u \cdot \nabla v \, d\mu$".]

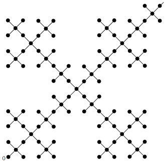
Examples (beyond the 1D interval)



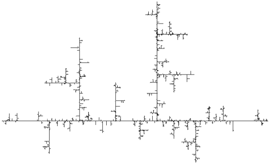
Sierpinski gasket



Sierpinski carpet



Vicsek tree



Random dendrite

1st-order calculus induced by Dirichlet forms

[Cipriani–Sauvageot '03, Hinz–Röckner–Teplyaev '13]
Let $\mathcal{H}$ denote the Hilbert space of L^2-vector fields associated with $(\mathcal{E}, \mathcal{F})$, defined by endowing $\mathcal{F} \otimes \mathcal{F}$ with the seminorm given as the bilinear extension of

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} := \frac{1}{2} \left[\mathcal{E}(abd, c) + \mathcal{E}(bcd, a) - \mathcal{E}(ac, bd) \right], \quad a, b, c, d \in \mathcal{F}.$$

modding out by zero seminorm elements and then completing.

Define a derivation $\partial : \mathcal{F} \to \mathcal{H}$ by $\partial f := f \otimes \mathbf{1}$.
Properties. Leibniz's rule: $\partial(fg) = (\partial f)g + f(\partial g)$. Energy: $\mathcal{E}(g, g) = \|\partial g\|^2_{\mathcal{H}}$.

We may extend ∂ to a densely defined operator $\partial : L^2(K, \mu) \to \mathcal{H}$ with domain $\mathcal{F}$.
Let $\mathcal{F}^*$ denote the dual space of $\mathcal{F}$.
Given $a \in \mathcal{H}$, define $(\partial^* a)(\varphi) := \langle a, \partial\varphi \rangle_{\mathcal{H}}$, $\varphi \in \mathcal{F}$.
This produces $\partial^* a \in \mathcal{F}^*$, and the linear map $\partial^* : \mathcal{F} \to \mathcal{F}^*$ satisfies $\|\partial^* a\|_{\mathcal{F}^*} \leq \|a\|_{\mathcal{H}}$.

Sobolev embedding: $\mathcal{F} \subset C(K) \subset L^2(K, \mu) \subset \mathcal{F}^*$

Example. For the classical Dirichlet form on $\Omega \subset \mathbb{R}^n$, $\mathcal{H}$ agrees with the space $L^2(\Omega, \mathbb{R}^n)$ of L^2-vector fields, and

$$\langle a \otimes b, c \otimes d \rangle_{\mathcal{H}} = \int_\Omega \langle b\nabla a, d\nabla c \rangle \, dx.$$

∂ and $-\partial^*$ can be id'd with the gradient and the divergence operators, resp.

Semilinear parabolic equations

Let B be a closed subset of K, $\Phi : \mathbb{R} \to \mathbb{R}$ be bounded Lipshitz with $\Phi(0) = 0$. Consider

$$\begin{cases} \partial_t u(t) = \sigma \Delta u(t) - \partial^* \left(\Phi(u(t)) V(t)\right) & \text{on } (0,T) \times K \setminus B, \\ u(0,\cdot) = u_0 & \text{on } K \setminus B, \\ u(t,\cdot)|_B = \bar{u}(t) & \text{on } (0,T) \times B. \end{cases}$$

Assumption. The Dirichlet problem with boundary condition on B has a unique solution, in the sense of Dirichlet forms.

Let $\mathcal{F}_0 = \{u \in \mathcal{F} : u|_B = 0\}$, $f \in L^2(0,T,\mathcal{F}_0^*)$, and $w_0 \in L^2(K,\mu)$.
Consider the corresponding problem with zero boundary condition.

$$\begin{cases} \partial_t w(t) = \sigma \Delta w(t) - \partial^* \left(\Psi(t, w(t)) V(t)\right) + f(t) & \text{on } (0,T) \times K \setminus B, \\ w(0,\cdot) = w_0 & \text{on } K \setminus B, \\ w(t,\cdot)|_B = 0 & \text{on } (0,T) \times B. \end{cases}$$

where $\Psi(t,v) := \Phi(v + h(t))$, and $h : (0,T) \times K \to \mathbb{R}$ is such that $h(t)$ is the unique harmonic extension of $\bar{u}(t)$ from B to K.

The operator of interest is $A(t,v) := -\sigma \Delta_{\mu,0} v + \partial^* \left(\Psi(t,v) V(t)\right)$, $v \in \mathcal{F}_0$.
(Here $\Delta_{\mu,0}$ denotes the Dirichlet Laplacian w.r.t. $L^2(K,\mu)$.)

Finite energy solutions: existence and uniqueness [C.–Hinz–Teplyaev '17+]

Consider the abstract Cauchy problem

$$\begin{cases} \partial_t w(t) + A(t, w(t)) = f(t) & \text{for a.e. } t \in (0,T), \\ w(0) = w_0 \end{cases} \tag{1}$$

Say that $w \in L^2(0,T,\mathcal{F}_0)$ with $\partial_t w \in L^2(0,T,\mathcal{F}_0^*)$ is a **(strong) solution** to (1) if the first identity holds in $\mathcal{F}_0^*$ for a.e. $t \in [0,T]$, and the second holds in $L^2(K,\mu)$.

Assumption A1. (K,R) is metrically doubling: $\exists C_D > 0$ such that any $B(x,2r)$ can be covered by C_D balls of radius r.
Assumption A2. μ is lower d-regular: $\exists C_L > 0$ such that $\mu(B(x,r)) \geq C_L r^d$ for any $x \in K$ and $r \in (0, \mathrm{diam} K)$.

Theorem (Existence)

Suppose $V \in L^\infty(0,T,\mathcal{H})$, and Assumptions A1 and A2 hold. Then there exists a solution to (1).

Theorem (Uniqueness)

Suppose $V \in L^2(0,T,\mathcal{H})$. Then there is at most one solution to (1).

Existence: proof sketch

We use the method of monotone operators as stated in [J.-L. Lions '65, Théorème 2.1].

Hypothesis (I): Coercivity $\Leftrightarrow$ A is a bounded demicontinuous operator.
This relies on the *a priori* estimate based on the Sobolev embedding:
If $h : (0, T) \to \mathcal{F}$ and $V \in L^\infty(0, T, \mathcal{H})$, then

$$|\langle \Psi(t, u)V(t), \partial v\rangle_{\mathcal{H}}| \leq C_R L_\Phi \|V\|_{L^\infty(0,T,\mathcal{H})} \mathcal{E}(u + h(t))^{1/2}\mathcal{E}(v)^{1/2}. \tag{2}$$

for any $u, v, \in \mathcal{F}_0$ and a.e. $t \in (0, T)$.

Hypothesis (II): A series of weak convergence conditions involving $A(u_k, \cdot)$, assuming that $u_k \rightharpoonup u$ in $L^2(0, T, \mathcal{F}_0)$.
This depends on (2) and the following estimates:

1. $\|\partial^* a\|_{\mathcal{F}^*} \leq \|a\|_{\mathcal{H}}, \quad a \in \mathcal{H}$
2. [Hinz–Rogers '16] Under Assumptions A1 and A2, $\exists C_1, C_2 > 0$ s.t. $\forall g \in \mathcal{F}$, $\forall a \in \mathcal{H}$, $\forall M > 0$,
$$\|ga\|^2_{\mathcal{H}} \leq \frac{1}{M}\mathcal{E}(g) + \left(C_1 M^d \|a\|^{2d}_{\mathcal{H}} + C_2\right) \|a\|^2_{\mathcal{H}} \|g\|^2_{L^2(K,\mu)}.$$

Uniqueness: proof sketch

Adapted from Landim–Mourragui–Sellami '00 and Bodineau–Lagouge '12, in conjunction with Dirichlet form theory

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Weak formulation: $\forall \varphi \in L^2(0, T, \mathcal{F}_0)$,

$$\int_0^T \int_K \partial_t u(t)\varphi(t)\, d\mu\, dt = -\int_0^T \mathcal{E}(u(t), \varphi(t))\, dt + \int_0^T \langle \Phi(u(t)+h(t))V(t), \partial\varphi(t)\rangle_{\mathcal{H}}\, dt.$$

If u^1 and u^2 are two weak solutions, let $\mathbf{u} := u^1 - u^2$.
We show contraction in time of $R_\delta \mathbf{u}$ in $L^1(K,\mu)$ norm, where
$R_\delta(y) = \dfrac{y^2}{2\delta}\mathbb{1}_{\{|y|<\delta\}} + \left(|y| - \dfrac{\delta}{2}\right)\mathbb{1}_{\{|y|\geq\delta\}}$ is a C^1 function tending to $|y|$ as $\delta \downarrow 0$.

Use integration by parts to show that for all $0 < t < t' < T$,

$$\int_K R_\delta(\mathbf{u}(t'))\, d\mu - \int_K R_\delta(\mathbf{u}(t))\, d\mu = \int_K \int_t^{t'} R'_\delta(\mathbf{u}(s))\partial_s \mathbf{u}(s)\, ds\, d\mu.$$

We then use the PDE to replace $\partial_s \mathbf{u}$ on the RHS, combine with basic properties (under scaling, under cutoffs [Markovian property]) and inequalities of Dirichlet forms to obtain that

$$\int_K |\mathbf{u}(t')|\, d\mu - \int_K |\mathbf{u}(t)|\, d\mu \leq 0$$

Regularity

Assume the operator A is time-independent ($A(t,v) = A(v)$), and $f \equiv 0$:

Set $\mathcal{D}(A_2) := \{v \in \mathcal{F}_0 : A(v) \in L^2(K,\mu)\}$ and $A_2 := A|_{\mathcal{D}(A_2)}$. Then under a "high-diffusivity assumption" (σ large enough), we can show that

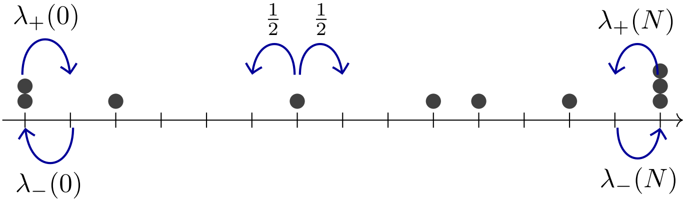
$$w(t) \in \mathcal{D}(A_2) \text{ for any } t \in (0,T), \text{ and } t\frac{dw}{dt}(t) \in L^\infty(0,T,L^2(K,\mu)).$$

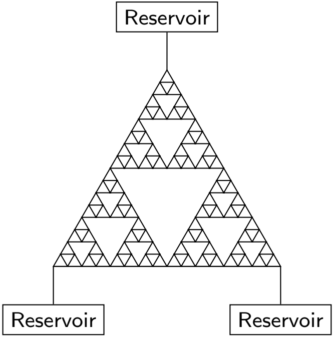
In fact we have $w(t) = S(t)w_0$, where $(S(t))_{t\geq 0}$ is the semigroup of (nonlinear) contractions on $L^2(K,\mu)$ generated by A_2.

We can show that the solutions are Lipschitz in time on any compact subinterval of $(0,T)$. However, it seems difficult to evaluate the improved spatial regularity, because the relationship between $\mathcal{D}(A_2)$ and $\mathcal{D}(\Delta_{\mu,0})$ is not clear, especially on singular spaces lacking a carré du champ.

Application: Hydrodynamic PDE arising from the exclusion process

- Particles are indistinguishable.
- In the bulk, $(0, N) \cap \mathbb{Z}$, particles undergo exclusion dynamics. (Random walks subject to the exclusion constraint: no two particles can occupy the same vertex.)
- At each boundary vertex $y \in \{0, N\}$, particles can be injected into or extracted from the bulk at resp. rate $\lambda_+(y)$ and $\lambda_-(y)$.
  - If $\lambda_+(y) = \lambda_-(y)$ for all y: system reaches equilibrium.
  - Otherwise: system is out of equilbrium (a mean density gradient develops between a "hot" reservoir and a "cold" one).





For large deviations purposes, we also introduce a drift in the nearest-neighbor jump rates ∂H for some $H \in C^1([0,T], \mathrm{dom}\Delta)$. The corresponding PDE reads

$$\begin{cases} \partial_t u(t) = \Delta u(t) - \partial^* \left(\chi(u(t))\partial H(t)\right) & \text{on } (0,T) \times K \setminus B, \\ u(0,\cdot) = u_0 & \text{on } K \setminus B, \\ u(t,\cdot)|_B = \bar{u}(t) & \text{on } (0,T) \times B. \end{cases}$$

Here $\chi(u) = u(1-u) \vee 0$, and B is a finite boundary set.
(B = Two endpoints in the case of the interval; three vertices of the triangle in the case of SG)

$\exists$ and ! of finite-energy sol'n to this PDE is a key input in the proof of the hydrodynamic limit.

Future directions, open problems

- As things stand, there appears to be a "gap" between the existence proof and the uniqueness proof.
  - $\exists$: Used Lions' monotone operator method, assumed some (natural) geometric conditions.
  - !: Estimate the difference of two solutions in $L^1(K, \mu)$ norm. Used properties of Dirichlet forms only; no dependence on the geometry.
- The uniqueness proof does not require the Dirichlet form to be local. Is locality of the resistance form needed for the existence proof? If not, it may be possible to extend this analysis to evolution equations with **non-local operators** (of jump type).
- Extracting more regularity information: Seems difficult using the monotone operator method. But again, semigroup methods which provide regularity results are not readily applicable to singular spaces.

Thank you!