

## Groupoid Methods in Wavelet Analysis

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*Dedicated to the memory of George W. Mackey*

ABSTRACT. We describe how the Deaconu-Renault groupoid may be used in the study of wavelets and fractals.

### 1. Introduction

This note serves two purposes. First, we want to describe investigations that we are undertaking which are inspired in large part by work of Palle Jorgensen and his collaborators, particularly Ola Bratteli, Dorin Dutkay and Steen Pedersen<sup>1</sup>. In their papers one finds a rich theory of wavelets on the one hand and topics in fractal analysis on the other. Further, the analysis in these papers is laced with representations of the Cuntz relations - finite families of isometries  $\{S_i\}_{i=1}^n$  such that  $\sum_{i=1}^n S_i S_i^* = 1$ . Very roughly speaking, these authors show that much of the analysis of wavelets and fractals that has appeared in recent years may be illuminated in terms of special representations of the Cuntz relations. Indeed, some of the most important advances are made by choosing an appropriate representation for these relations. Our motivation was to understand the extent to which the use of the Cuntz relations is *intrinsic* to the situation under consideration. We wanted to separate intrinsically occurring representations of the Cuntz relations from those that are imposed by special choices. We hoped, thereby, to clarify the degrees of freedom that go into the representations found in the work we are discussing.

As it turns out, the Cuntz isometries that arise in the work of Jorgensen et. al. may be expressed in terms of representations of the Deaconu-Renault groupoid associated to an appropriate local homeomorphism of a compact Hausdorff space. Our second purpose is to show how the  $C^*$ -algebra of this groupoid is related to a number of other  $C^*$ -algebras that one can attach to a local homeomorphism. In particular, we show that the  $C^*$ -algebra may be realized as a Cuntz-Pimsner

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<sup>1</sup>A number of relevant papers are cited in the references to this paper, but for a more comprehensive list, the reader should consult the books [3] and [21].

algebra in two different ways and that, in general, it is a quotient of certain other  $C^*$ -algebras that one may build from the local homeomorphism.

Recall that a wavelet is usually understood to be a vector  $\psi$  in  $L^2(\mathbb{R})$  such that the family

$$\{D^j T^k \psi : j, k \in \mathbb{Z}\}$$

is an orthonormal basis for  $L^2(\mathbb{R})$ , where  $T$  is the operator of translation by 1, i.e.,  $T\xi(x) = \xi(x - 1)$ ,  $\xi \in L^2(\mathbb{R})$ , and where  $D$  is dilation by 2, i.e.,  $D\xi(x) = \sqrt{2}\xi(2x)$ . One of the principal problems in the study of wavelets is to construct them with various pre-assigned properties. That is, one wants to “tune” the parameters that enter into the analysis of wavelets so that the wavelet one constructs exhibits the prescribed properties. So one’s first task is to identify those parameters and to understand the relations among them.

Fractals, on the other hand, are spaces that possess some sort of scaling. That is, as is customarily expressed, fractals exhibit the same features at all scales. How to make this statement precise and how to construct such spaces in useful ways are, of course, the objects of considerable research. Most of the known examples of fractals are closely connected to spaces endowed with a local homeomorphism that is not injective. This may seem like a banal oversimplification, but reflection on it does lead to natural representations of the Cuntz relations, as we shall see, that are intrinsic to the geometry of the situation. Since wavelets have a natural scaling built into them, it is natural to contemplate the possibility of building natural wavelet-like orthonormal bases in  $L^2$ -spaces erected on fractals. This is indeed possible, and much of the work by Jorgensen and his co-authors has been devoted to realizing the possibilities.

Our contribution is to observe that the Deaconu-Renault groupoid associated with a local homeomorphism of a compact Hausdorff space provides a natural environment in which to set up fractal analysis, and that the  $C^*$ -algebras of the groupoids carry natural, geometrically induced families of isometries. The representations that Jorgensen and his collaborators study come from representations of this groupoid. Further, wavelets and other orthonormal bases on the fractals are seen to be artifacts of the representation theory of the groupoid. In short, groupoids help to clarify constructions of both fractals and wavelets and help to analyze the parameters involved.

## 2. The Setup

Throughout this note,  $X$  will denote a fixed compact Hausdorff space and  $T : X \rightarrow X$  will denote a surjective local homeomorphism. One can relax these hypotheses in various ways and in various situations, but we shall not explore the possibilities here. The principal examples to keep in mind are the following.

**EXAMPLE 2.1.** Let  $X$  be the circle or torus  $\mathbb{T}$ . The local homeomorphism in this case is also an endomorphism of the abelian group structure on  $\mathbb{T}$ :  $T(z) = z^N$ ,  $z \in \mathbb{T}$ , where  $N$  is a natural number. The case when  $N = 2$ , provides a link to “classical” wavelets.

**EXAMPLE 2.2.** Let  $A$  be an  $n \times n$  dilation matrix. That is, suppose  $A$  has integer entries and that the determinant of  $A$  has absolute value  $d$  (which must be a positive integer) that is greater than 1. If we view the  $n$ -torus,  $\mathbb{T}^n$ , as the quotient group  $\mathbb{R}^n/\mathbb{Z}^n$ , then  $A$  induces a local homeomorphism  $T : \mathbb{T}^n \rightarrow \mathbb{T}^n$  via

the formula  $T(x + \mathbb{Z}^n) = Ax + \mathbb{Z}^n, x + \mathbb{Z}^n \in \mathbb{T}^n$ . It is not difficult to see that  $T$  is  $d$ -to-1.

EXAMPLE 2.3. In this example we connect our discussion to the theory of iterated function systems, which is one of the main ways to construct fractals [1]. Assume  $X$  is a compact space endowed with a metric,  $d$  say, and let  $(\varphi_1, \dots, \varphi_n)$  be a system of maps on  $X$  for which there are constants  $c_1$  and  $c_2$  such that  $0 < c_1 \leq c_2 < 1$  and such that  $c_1 d(x, y) \leq d(\varphi_i(x), \varphi_i(y)) \leq c_2 d(x, y)$  for each  $i$ . Then each  $\varphi_i$  is homeomorphism onto its range. Also, the family  $(\varphi_1, \dots, \varphi_n)$  induces a map  $\Phi$  on the space of non-empty closed (and hence compact) subsets  $K$  of  $X$  via the formula

$$\Phi(K) = \cup_{i=1}^n \varphi_i(K).$$

It is then easy to see that  $\Phi$  is a strict contraction in the Hausdorff metric on the space of nonempty closed subsets of  $X$  and so there is a unique nonempty compact subset  $K$  of  $X$  such that  $\Phi(K) = K$ . This  $K$  is called *the* invariant subset of the system. It is the fractal associated with the system. We shall assume that  $X$  is the invariant set. It is important to note that there may be overlap between  $\varphi_i(X)$  and  $\varphi_j(X)$  for  $i \neq j$ . Consequently, the  $\varphi_i$  need not be branches of the inverse of a local homeomorphism. One way to “get around” this limitation is to *lift* the system in the sense of [1, Page 155]. For this purpose, let  $E^\infty$  the space of infinite words over the alphabet  $E = \{1, \dots, n\}$ . Then in the product topology  $E^\infty$  is compact and we can give  $E^\infty$  a complete metric such that the maps  $\sigma_i : E^\infty \rightarrow E^\infty$  defined by the formula  $\sigma_i(w) = (i, w_1, w_2, \dots)$ , where  $w = (w_1, w_2, \dots)$ , are contractions of the same type as the  $\varphi_i$ . The iterated function system on  $X \times E^\infty$ ,  $(\tilde{\varphi}_1, \dots, \tilde{\varphi}_n)$ , defined by the formula

$$\tilde{\varphi}_i(x, w) = (\varphi_i(x), \sigma_i(w))$$

then has a unique nonempty closed invariant subset  $\tilde{X}$  of  $X \times E^\infty$ . That is  $\cup_{i=1}^n \tilde{\varphi}_i(\tilde{X}) = \tilde{X}$ . The system  $(\tilde{\varphi}_1, \dots, \tilde{\varphi}_n)$  on  $\tilde{X}$  is called the *lifted system*. The ranges of the  $\tilde{\varphi}_i$  are disjoint and so there is a local homeomorphism  $T$  of  $\tilde{X}$  such that the  $\tilde{\varphi}_i$  are the branches of the inverse of  $T$ . As is discussed in Section 4.6 of [1], the systems  $(\varphi_1, \dots, \varphi_n)$  and  $(\tilde{\varphi}_1, \dots, \tilde{\varphi}_n)$  share many features in common and, from some points of view, are interchangeable.

### 3. The Deaconu-Renault Groupoid

The Deaconu-Renault groupoid associated with the local homeomorphism  $T : X \rightarrow X$  is

$$G = \{(x, n, y) \in X \times \mathbb{Z} \times X : T^k(x) = T^l(y), n = k - l\}.$$

Two triples  $(x_1, n_1, y_1)$  and  $(x_2, n_2, y_2)$  are composable if and only if  $x_2 = y_1$  and in that case,  $(x_1, n_1, y_1)(x_2, n_2, y_2) = (x_1, n_1 + n_2, y_2)$ . The inverse of  $(x, n, y)$  is  $(y, -n, x)$ . A basis for the topology on  $G$  is given by the sets

$$Z(U, V, k, l) := \{(x, k - l, y) \in G : x \in U, y \in V\},$$

where  $U$  and  $V$  are open subsets of  $X$  such that  $T^k|U, T^l|V$  are homeomorphisms and  $T^k(U) = T^l(V)$ . Thus  $Z(U, V, k, l)$  is essentially the graph of  $(T^l|V)^{-1} \circ (T^k|U)$  and is a  $G$ -set in the sense of Renault [32]. The  $G$ -sets form a pseudogroup  $\mathfrak{G}$ , viz. the pseudogroup of partial homeomorphisms generated by  $T$ . The sets  $Z(U, V, k, l)$  form a basis for  $\mathfrak{G}$ . The groupoid  $G$  is (isomorphic and homeomorphic to) the groupoid of germs of  $\mathfrak{G}$  precisely when the local homeomorphism  $T$  is essentially

free, meaning that for no  $m$  and  $n$  does  $T^m = T^n$  on any open subset of  $X$  [34, Proposition 2.8].

The groupoid  $G$  is  $r$ -discrete or étale and so admits a Haar system of counting measures. Consequently, we may define a  $*$ -algebra structure on  $C_c(G)$  as follows. For  $f, g \in C_c(G)$  we set

$$f * g(x, k - l, y) = \sum f(x, m - n, z) \cdot g(z, (n + k) - (m + l), y),$$

where the sum ranges over all  $m, n,$  and  $z$  such that  $T^m x = T^n z,$  and  $T^{n+k} z = T^{m+l} y,$  and we define

$$f^*(x, k - l, y) = \overline{f(y, l - k, x)}.$$

The algebra  $C_c(G)$  can be completed to form a  $C^*$ -algebra, denoted  $C^*(G),$  in the norm

$$\|f\| := \sup \|\pi(f)\|$$

where the supremum is taken over all  $*$ -homomorphisms of  $C_c(G)$  into  $B(H_\pi)$  that are continuous with respect to the inductive limit topology on  $C_c(G)$  and the weak operator topology on  $B(H_\pi),$  the algebra of operators on the Hilbert space of  $\pi, H_\pi.$  We will discuss the representations of  $C_c(G)$  more fully later, but first we want to call attention to some special clopen relations “on”  $X.$

For fixed positive integers  $m$  and  $n,$  we set  $R_{n,m} := \{(x, n - m, y) \in G : T^n x = T^m y\}.$  Evidently,  $R_{n,m}$  is a union of the basic sets  $Z(U, V, m, n),$  and so is open in  $G.$  It is also closed, since its complement is open by virtue of being a union of sets of the form  $Z(U, V, k, l),$  with  $(k, l) \neq (m, n).$  The sets  $R_{n,m},$  with  $m = n,$  are of special importance:  $R_{0,0}$  may be identified with the diagonal  $\Delta$  in  $X \times X,$  while for  $k > 0,$   $R_{k,k}$  may be identified with the relation  $X *_T X := \{(x, y) : T^k(x) = T^k(y)\}$  in  $X \times X.$  The  $C^*$ -algebra of  $R_{k,k}, C^*(R_{k,k}),$  which may be identified with the closure of  $C_c(R_{k,k})$  in  $C^*(G),$  is the cross sectional  $C^*$ -algebra of a matrix bundle over  $X$  and, therefore, is a continuous trace  $C^*$ -algebra. (See [24] for a discussion of algebras of the form  $C^*(R_{k,k}).$ ) The sequence of inclusions  $R_{0,0} \subset R_{1,1} \subset R_{2,2} \subset \dots$  leads to the sequence of inclusions  $C^*(R_{k,k}) \subset C^*(R_{k+1,k+1}), k = 0, 1, 2, \dots,$  and, consequently, we see that if  $R_\infty = \{(x, 0, y) : T^n x = T^n y \text{ for some } n\} = \bigcup R_{n,n},$  then  $C^*(R_\infty)$  is the inductive limit  $\varinjlim C^*(R_{n,n}).$  We note that  $R_\infty$  is the kernel of the fundamental homomorphism on  $G: (x, n, y) \rightarrow n,$  which implements the gauge automorphism group  $\{\gamma_z\}_{z \in \mathbb{T}}$  defined on  $C_c(G)$  by the formula  $\gamma_z(f)(x, n, y) = z^n f(x, n, y).$  The algebra  $C^*(R_\infty)$  is the fixed point algebra of  $\{\gamma_z\}_{z \in \mathbb{T}},$  also known as the *core* of  $C^*(G).$  For these things, and more, we refer the reader to [8, 34].

It is a straightforward calculation, performed first by Deaconu [6], to see that the local homeomorphism  $T$  on  $X$  induces a  $*$ -endomorphism  $\alpha : C^*(R_\infty) \rightarrow C^*(R_\infty)$  defined by the equation,

$$(1) \quad \alpha(f)(x, 0, y) = \frac{1}{\sqrt{|T^{-1}(Tx)||T^{-1}(Ty)|}} f(Tx, 0, Ty),$$

$f \in C_c(R_\infty).$  Further, a similar calculation shows that the function  $S$  in  $C_c(G)$  defined by the equation

$$(2) \quad S(x, m - n, y) = \begin{cases} \frac{1}{\sqrt{|T^{-1}(Tx)|}}, & \text{if } m = 1, n = 0, Tx = y, \\ 0 & \text{otherwise,} \end{cases}$$

is an isometry that implements  $\alpha$  in the sense that

$$(3) \quad \alpha(f) = SfS^*,$$

$f \in C_c(R_\infty)$ . In particular, observe that

$$(4) \quad SS^*(x, k-l, y) = \frac{1}{|T^{-1}(Tx)|} 1_{R_{1,1}}(x, k-l, y).$$

As we shall see,  $S$  is the source of all the isometries in the papers by Jorgensen et al. It is an intrinsic feature of the  $C^*$ -algebra that comes from the basic data:  $X$  and the local homeomorphism  $T$ . In fact, we have the following theorem, Theorem 3.1, that makes precise the assertion that  $C^*(G)$  is the universal  $C^*$ -algebra generated by  $C^*(R_\infty)$ ,  $\alpha$ , and  $S$ . In fact, there are several different perspectives from which to see how  $C^*(G)$  is constructed from the space  $X$  and local homeomorphism  $T$ . We want to examine these and to compare them with various approaches in the literature. Therefore the proof will be given after further discussion.

**THEOREM 3.1.** *Let  $\tilde{\pi} : C^*(G) \rightarrow \mathcal{B}(H)$  be a  $C^*$ -representation. Define  $\pi : C^*(R_\infty) \rightarrow \mathcal{B}(H)$  by  $\pi = \tilde{\pi}|_{C^*(R_\infty)}$ , and let  $S_+ = \tilde{\pi}(S)$ . Then*

- (1)  $\pi(\alpha(f)) = S_+ \pi(f) S_+^*$ ; and
- (2)  $\pi(\mathcal{L}(f)) = S_+^* \pi(f) S_+$ , where  $\mathcal{L}(f) = S^* f S$  is the transfer operator associated with  $\alpha$ ,

$$(5) \quad \mathcal{L}(f)(x, 0, y) = \frac{1}{\sqrt{|T^{-1}(x)||T^{-1}(y)|}} \sum_{\substack{Tu=x \\ Tv=y}} f(u, 0, v).$$

Conversely, given  $(\pi, S_+)$ , where  $\pi : C^*(R_\infty) \rightarrow \mathcal{B}(H)$  is a  $C^*$ -representation and  $S_+$  is an isometry on  $H$  such that 1. and 2. are satisfied, then there is a unique representation  $\tilde{\pi} : C^*(G) \rightarrow \mathcal{B}(H)$  such that  $\tilde{\pi}(f) = \pi(f)$  for all  $f \in C^*(R_\infty)$  and  $\tilde{\pi}(S) = S_+$ .

Recall, next, that if  $A$  is a  $C^*$ -algebra, then a  $C^*$ -correspondence over  $A$  is an  $A$ - $A$ -bimodule  $E$  such that  $E_A$  is a Hilbert  $C^*$ -module and the left action is given by a  $C^*$ -homomorphism  $\phi$  from  $A$  into the bounded adjointable operators on  $E$  [28],  $\mathcal{L}(E)$ . We write  $\mathcal{K}(E)$  for the space of compact operators on  $E$ , i.e.,  $\mathcal{K}(E)$  is the closed linear span of the operators  $\xi \otimes \eta^*$ ,  $\xi, \eta \in E$ , defined by the formula  $\xi \otimes \eta^*(\zeta) := \xi \langle \eta, \zeta \rangle$ , and we write  $J$  for the ideal  $\phi^{-1}(\mathcal{K}(E))$  in  $A$ . A Cuntz-Pimsner covariant representation of  $E$  in a  $C^*$ -algebra  $B$  is a pair  $(\pi, \psi)$ , where  $\pi$  is a  $C^*$ -representation of  $A$  in  $B$  and  $\psi$  is a map from  $E$  into  $B$  such that

- (1)  $\psi(\phi(a)\xi b) = \pi(a)\psi(\xi)\pi(b)$ , for all  $a, b \in A$  and all  $\xi \in E$ ;
- (2) for all  $\xi, \eta \in E$ ,  $\psi(\xi)^* \psi(\eta) = \pi(\langle \xi, \eta \rangle)$ ; and
- (3) for all  $a \in J$ ,  $(\psi, \pi)^{(1)}(\phi(a)) = \pi(a)$ , where  $(\psi, \pi)^{(1)}$  is the representation of  $\mathcal{K}(E)$  in  $B$  defined by the formula  $(\psi, \pi)^{(1)}(\xi \otimes \eta^*) = \psi(\xi)\psi(\eta)^*$ ,  $\xi \otimes \eta^* \in \mathcal{K}(E)$ .

There is a  $C^*$ -algebra  $\mathcal{O}(E)$  and Cuntz-Pimsner representation  $(k_A, k_E)$  of  $E$  in  $\mathcal{O}(E)$  that is universal for all Cuntz-Pimsner representations of  $E$ . That is, if  $(\pi, \psi)$  is a Cuntz-Pimsner representation of  $E$  in a  $C^*$ -algebra  $B$ , then there is a unique  $C^*$ -representation  $\rho$  of  $\mathcal{O}(E)$  in  $B$  such that  $\rho \circ k_A = \pi$  and  $\rho \circ k_E = \psi$ . The representation  $\rho$  is often denoted  $\pi \times \psi$ . This was proved essentially by Pimsner in [31] and in the form stated here in [18, Proposition 1.3].

DEFINITION 3.2. The *Deaconu  $C^*$ -correspondence*  $\mathcal{X}$  over the  $C^*$ -algebra  $C(X)$  is the completion of  $C(X)$  under the inner product

$$\langle \xi, \eta \rangle(x) = \frac{1}{|T^{-1}(x)|} \sum_{Ty=x} \overline{\xi(y)}\eta(y),$$

with the left and right actions of  $C(X)$  given by  $(a \cdot \xi \cdot b)(x) = a(x)\xi(x)b(Tx)$ .

The definition we have given is slightly different from the one given in [7]. He does not divide by  $|T^{-1}(x)|$ . However, it is easy to see that the two  $C^*$ -correspondences are isomorphic. The following theorem is due to Deaconu [7, Propositions 3.1 and 3.3]. The formulation we present is that of [8, Theorem 7], which is slightly more general. The proof in [8] is based on the gauge invariant uniqueness theorem found in [18, Theorem 4.1].

THEOREM 3.3. (Deaconu) Define  $\iota : C(X) \rightarrow C^*(G)$ , by the equation

$$\iota(\varphi)(x, k - l, y) = \varphi(x)1_{R_{0,0}}(x, k - l, y),$$

and  $\psi : \mathcal{X} \rightarrow C^*(G)$ , by the equation  $\psi(\xi) = \iota(\xi)S$ . Then  $(\iota, \psi)$  is a faithful Cuntz-Pimsner covariant representation of  $(C(X), \mathcal{X})$  in  $C^*(G)$ , whose image generates  $C^*(G)$  and gives an isomorphism between  $C^*(G)$  and  $\mathcal{O}(\mathcal{X})$ .

In [17], Exel introduced a crossed product associated to an endomorphism  $\alpha$  of a  $C^*$ -algebra  $A$  and transfer operator  $\mathcal{L}$  for  $\alpha$ . That is,  $\mathcal{L}$  is a positive operator on  $A$  that satisfies the equation  $\mathcal{L}(a\alpha(b)) = \mathcal{L}(a)b$  for all  $a, b \in A$ . Exel's crossed product, denoted  $A \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  can also be described as a relative Cuntz-Pimsner algebra, as was accomplished by Brownlowe and Raeburn in [4]. We adopt their perspective and assume also that  $A$  is unital, but we don't assume that  $\alpha$  is unital. Let  $M_{\mathcal{L}}$  denote the completion of  $A$  in the inner product  $\langle a, b \rangle := \mathcal{L}(a^*b)$ , and give  $M_{\mathcal{L}}$  the right and left actions of  $A$  defined by the formulae  $m \cdot a := m\alpha(a)$  and  $a \cdot m = \phi(a)m = am$ . As a left  $A$ -module  $M_{\mathcal{L}}$  is cyclic and the image of 1 in  $M_{\mathcal{L}}$  is a cyclic vector, which we denote by  $\xi_0$ . If  $(\pi, \psi)$  is a Cuntz-Pimsner representation of  $M_{\mathcal{L}}$  in a  $C^*$ -algebra  $B$ , then the image of  $\xi_0$  in  $B$ ,  $\psi(\xi_0)$ , is an isometry  $V$ , say. Then  $(\pi, \psi)$  is completely determined by  $\pi$  and  $V$  in the following sense: Let  $\pi$  be a representation of  $A$  in a  $C^*$ -algebra  $B$ , let  $V$  be an isometry in  $B$ , and define  $\psi : M_{\mathcal{L}} \rightarrow B$  by the formula,  $\psi(\phi(a)\xi_0) = \pi(a)V$ , then  $(\pi, \psi)$  is a Cuntz-Pimsner representations of  $M_{\mathcal{L}}$  in  $B$  if and only if the following equations CP1., CP2. and CP3. are satisfied:

- CP1.  $V\pi(a) = \pi(\alpha(a))V$  for all  $a \in A$ ;
- CP2.  $V^*\pi(a)V = \pi(\mathcal{L}(a))$  for all  $a \in A$ ; and
- CP3.  $\pi(a) = (\psi, \pi)^{(1)}(\phi(a))$ , for all  $a \in J$ .

THEOREM 3.4. In the context of our groupoid,  $G$ , let  $A = C^*(R_{\infty})$ , let  $\alpha$  be the endomorphism of  $A$  defined by equation (1), let  $\mathcal{L}$  be the associated transfer operator (5) and let  $M_{\mathcal{L}}$  be the correspondence over  $A$  defined by Brownlowe and Raeburn that we just described. Then the identity representation  $\iota$  mapping  $C^*(R_{\infty})$  into  $C^*(G)$  together with the isometry  $S$  defined by equation (2), determine a Cuntz-Pimsner representation  $(\iota, \psi)$  of  $M_{\mathcal{L}}$  in  $C^*(G)$  that implements an isomorphism of  $\mathcal{O}(M_{\mathcal{L}})$  onto  $C^*(G)$ .

PROOF. Equation CP.1 follows from equation (3) and equation CP.2, which is the same as the second equation of Theorem 3.1, is a straightforward calculation. We need to verify equation CP.3. Since  $\xi_0$  is a cyclic vector for the left action of  $A$  on  $M_{\mathcal{L}}$ ,  $\mathcal{K}(M_{\mathcal{L}})$  is the closed linear span of elements of the form  $\phi(a)\xi_0 \otimes \xi_0^*\phi(b)$ ,

where  $a$  and  $b$  range over  $A$ . So, if  $\phi(a)$  is compact, there is a sequence whose terms are of the form  $\sum_i \phi(a_i)\xi_0 \otimes \xi_0^*\phi(b_i)$  that converges to  $\phi(a)$  in  $\mathcal{K}(M_{\mathcal{L}})$ . So, if we apply  $\iota(a)$  to an element of the form  $\iota(b)S = \psi(\phi(b)\xi_0)$ , then we may write the following equation

$$\begin{aligned}
 (6) \quad \iota(a)\iota(b)S &= \psi(\phi(a)\phi(b)\xi_0) = \lim \psi\left(\sum_i \phi(a_i)\xi_0 \otimes \xi_0^*\phi(b_i)(\phi(b)\xi_0)\right) \\
 &= \lim \sum_i \iota(a_i)S S^* \iota(b_i)\iota(b)S \\
 &= \lim \sum_i \iota(a_i)S \iota(\mathcal{L}(b_i b)) \\
 &= \lim \sum_i \iota(a_i)\iota(\alpha \circ \mathcal{L}(b_i b))S \\
 &= \lim \sum_i (\psi, \iota)^{(1)}(\phi(a_i)\xi_0 \otimes \xi_0^*\phi(b_i))(\iota(b)S) \\
 &= (\psi, \iota)^{(1)}(\phi(a))(\iota(b)S).
 \end{aligned}$$

By [18, Lemma 4.4.1]

$$(7) \quad \psi(\xi \otimes \eta^*(\phi(b)\xi_0)) = (\psi, \iota)^{(1)}(\xi \otimes \eta^*)(\iota(b)S),$$

which shows that for all  $T \in \mathcal{K}(M_{\mathcal{L}})$ ,  $(\psi, \iota)^{(1)}(T)$  is determined by its values on elements of the form  $\iota(b)S$ . Thus, equations (6) and (7) together show that if  $a \in J$ , then  $\iota(a) = (\psi, \iota)^{(1)}(\phi(a))$ . Thus  $(\iota, \psi)$  is a Cuntz-Pimsner representation, the range of which clearly generates  $C^*(G)$ . So all we need to show is that  $\iota \times \psi$  is injective. But this is immediate from the injectivity of  $\iota$ , by the gauge-invariant uniqueness theorem [18, Theorem 4.1].  $\square$

PROOF OF THEOREM 3.1. The fact that conditions 1. and 2. of the theorem are satisfied is an easy calculation. The ‘‘converse’’ assertion follows from Theorem 3.4 because, as is easily seen, if  $(\pi, S_+)$  are given, acting on a Hilbert space  $H$ , say, then we obtain a Cuntz-Pimsner representation  $(\pi, \psi)$  of  $M_{\mathcal{L}}$  by setting  $\psi(\xi) := \pi(\xi)S_+$ . This representation ‘‘integrates’’ to give a  $C^*$ -representation of  $\mathcal{O}(\mathcal{X})$ , which by Theorem 3.3 is  $C^*(G)$ .  $\square$

#### 4. Filter Banks

DEFINITION 4.1. A family  $\{m_i\}_{i=1, \dots, N} \subseteq \mathcal{X}$  is called a *filter bank* if it is an orthonormal basis for  $\mathcal{X}$ .

This means that  $\langle m_i, m_j \rangle = 0$  if  $i \neq j$ , and  $\langle m_i, m_i \rangle = 1$ . Note that this last condition is much stronger than asserting that each  $m_i$  has norm 1. In general a module  $\mathcal{X}$  need not have an orthonormal basis. Even some modules built on  $\mathbb{T}^n$  with the map  $z \rightarrow Az$  may fail to have orthonormal bases. However, on  $\mathbb{T}^1$  they exist.

DEFINITION 4.2. If  $\{m_i\}_{i=1, \dots, N}$  is a filter bank, we call  $m_1$  the *low pass filter* and the rest *high pass filters*.

One problem of great importance is to decide when a function  $m$  in  $\mathcal{X}$  satisfying  $\langle m, m \rangle = 1$  can be completed to an orthonormal basis, i.e., when can such a function  $m$  be viewed as a low pass filter in a filter bank. This depends to a great extent

upon the underlying geometry of the situation under consideration, as Packer and Rieffel have shown [29, 30].

We note, too, that while we have been emphasizing the topological situation, there is a Borel version of our analysis. In this situation Borel orthonormal bases always exist and low pass filters can be completed to a filter bank.

**THEOREM 4.3.** *Define  $\beta : C(X) \rightarrow C(X)$  by  $\beta(f) = f \circ T$ ,  $f \in C(X)$ , and adopt the notation of Theorem 3.3. The following assertions are valid in  $C^*(G)$  :*

- (1)  $\iota(\beta(a))S = S\iota(a)$ , for  $a \in C(X)$ .
- (2) If  $\{m_1, \dots, m_n\}$  is a filter bank and if  $S_i := \psi(m_i)$ , then  $\{S_i\}$  is a Cuntz family of isometries in  $C^*(G)$  such that

$$\iota(\beta(a))S_i = S_i\iota(a).$$

- (3) For all  $a \in C(X)$

$$(8) \quad \iota(\beta(a)) = \sum_{i=1}^n S_i\iota(a)S_i^*.$$

The proof of Theorem 4.3 is a straightforward calculation and so will be omitted. Nevertheless, there are several useful points to be raised about the result.

Suppose, quite generally, that  $A$  is a  $C^*$ -algebra and that  $\alpha$  is an endomorphism of  $A$ . Then the powers of  $\alpha$  can be used to build an inductive system  $(\{A_n\}_{n=0}^\infty, \{\alpha_{m,n}\}_{m \geq n})$  in a familiar fashion: one takes  $A_n$  to be  $A$  for every  $n$  and sets  $\alpha_{m,n} := \alpha^{m-n}$ , when  $m \geq n$ . The inductive limit of this system,  $A_\infty$ , exists, but may be zero. In the event the limit is not 0, then, as Stacey proves in Proposition 3.2 of [37], there is, for each positive integer  $n$ , a unique  $C^*$ -algebra  $B$  and a pair  $(\iota, \{t_i\}_{i=1}^n)$  consisting of a  $*$ -homomorphism  $\iota : A \rightarrow B$  such that  $\bar{\iota}(1_{M(A)}) = 1_{M(B)}$ , where  $\bar{\iota}$  denotes the extension of  $\iota$  to the multiplier algebra of  $A$ ,  $M(A)$ , and a family of isometries in the multiplier algebra of  $B$ ,  $\{t_i\}_{i=1}^n \subseteq M(B)$ , such that

- (1)  $\{t_i\}_{i=1}^n$  is a Cuntz family of isometries, i.e.,  $t_i^*t_j = \delta_{ij}1_{M(B)}$ , for  $i, j = 1, 2, \dots, n$ , and  $\sum_{i=1}^n t_i t_i^* = 1_{M(B)}$ . When  $n = 1$ ,  $t = t_1$  is simply an isometry.
- (2) For all  $a \in A$ ,  $\iota(\alpha(a)) = \sum_{i=1}^n t_i \iota(a) t_i^*$ .
- (3) If  $(\pi, \{T_i\}_{i=1}^n)$  is a family consisting of a  $C^*$ -representation of  $A$  on a Hilbert space  $H$  and a Cuntz family of isometries  $\{T_i\}_{i=1}^n$  in  $B(H)$ , then there is a nondegenerate representation  $(\pi \times T)$  of  $B$  on  $H$  so that  $(\pi \times T) \circ \iota = \pi$  and  $(\pi \times T)(t_i) = T_i$ ,  $i = 1, 2, \dots, n$ . (The family  $(\pi, \{T_i\}_{i=1}^n)$  is called a *Cuntz-covariant representation of order  $n$*  of the system  $(A, \alpha)$ .)
- (4)  $B$  is the  $C^*$ -algebra generated by  $\iota(A)$  and elements of the form  $\iota(a)t_i$ ,  $i = 1, 2, \dots, n$  and  $a \in A$ .

**DEFINITION 4.4.** The  $C^*$ -algebra  $B$  just described is called *the Stacey crossed product of order  $n$  determined by  $A$  and  $\alpha$* , and is denoted  $A \rtimes_n^\alpha \mathbb{N}$ .

Note that when  $n = 1$ , the endomorphism in a Stacey crossed product of order 1 cannot be unital if the embedding  $\iota$  is injective. This happens if and only if there is a Cuntz-covariant representation  $(\pi, T)$  of order 1 with a faithful  $\pi$ . In the setting of Theorem 3.1, it is clear that  $\alpha$  is not unital by virtue of equation (3). Also, by virtue of equation 1. in the statement of that theorem it is natural to speculate about the relation between  $C^*(G)$  and the Stacey crossed product of



order 1 determined by  $C^*(R_\infty)$  and  $\alpha$ . It turns out that the crossed product that Exel would associate to  $C^*(R_\infty)$ ,  $\alpha$ , and  $\mathcal{L}$ , in [17] and which he would denote by  $C^*(R_\infty) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ , is isomorphic to  $C^*(R_\infty) \rtimes_1^\alpha \mathbb{N}$  by his [17, Theorem 4.7]. On the other hand, Brownlowe and Raeburn show that Exel's algebra  $C^*(R_\infty) \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$  is isomorphic to the relative Cuntz-Pimsner algebra determined the ideal  $\overline{A\alpha(A)A} \cap J$ , where  $A = C^*(R_\infty)^2$ . Now in this situation  $J$  coincides with  $A$  because  $\phi(1) = \phi(P)$ , where  $P = SS^*$ , and because  $\phi(P) = \xi_0 \otimes \xi_0^*$ . On the other hand, the ideal  $\overline{A\alpha(A)A}$  is proper. Thus, the relative Cuntz-Pimsner algebra determined by  $\overline{A\alpha(A)A} \cap J$  has the Cuntz-Pimsner algebra  $\mathcal{O}(M_{\mathcal{L}})$  as a proper quotient, by [18, Proposition 3.14]. So, in our setting, we see that  $C^*(G)$  is a proper quotient of  $C^*(R_\infty) \rtimes_{\alpha, \mathcal{L}} \mathbb{N} \simeq C^*(R_\infty) \rtimes_1^\alpha \mathbb{N}$ . On the other hand, Theorem 4.3 suggests that  $C^*(G)$  may be the Stacey crossed product  $C(X) \rtimes_n^\beta \mathbb{N}$ , but we are unable to determine the precise circumstances under which this may happen. Nevertheless, as Theorem 4.3 shows,  $C^*(G)$  contains a Cuntz covariant representation of order  $n$  of  $(C(X), \beta)$ , and therefore any  $C^*$ -representation of  $C^*(G)$  produces automatically a Cuntz-covariant representation of  $(C(X), \beta)$ . These are the starting point of Bratteli and Jorgensen's analysis [2, Proposition 1.1].

### 5. Representations of $C^*(G)$

Renault worked out the structure theory of the most general representation of any groupoid  $C^*$ -algebra in [33]. We discuss here certain aspects of it in our special setting that is relevant for applications to wavelets. Let  $\pi : C^*(G) \rightarrow \mathcal{B}(\mathcal{H})$  be a  $C^*$ -representation, where  $G$  for the moment is an arbitrary locally compact groupoid with Haar system  $\{\lambda^u\}_{u \in G^{(0)}}$ . Then  $\pi$  determines and is determined by a triple  $(\mu, \mathcal{H}, U)$ , where  $\mu$  is a quasi-invariant measure on  $G^{(0)} = X$ ;  $\mathcal{H}$  is a (Borel) Hilbert bundle on  $X$ , and  $U$  is a representation of  $G$  on  $\mathcal{H}$ . The relation between  $\pi$  and the triple  $(\mu, \mathcal{H}, U)$  is expressed through the equation

$$\pi(f)\xi(u) = \int_{G^u} f(\gamma)(U(\gamma)\xi(s(\gamma)))\Delta^{\frac{1}{2}}(\gamma) d\lambda^u(\gamma),$$

where  $\xi$  is an  $L^2(\mu)$ -section of the bundle  $\mathcal{H}$  and  $\Delta$  is the modular function of the measure  $\mu$ . In more detail, let  $\nu = \int_{G^{(0)}} \lambda^u d\mu(u)$  and let  $\nu^{-1}$  be the image of  $\nu$  under inversion. Then to say  $\mu$  is quasi-invariant is to say that  $\nu$  and  $\nu^{-1}$  are mutually absolutely continuous. In this case,  $\Delta$  is defined to be  $\frac{d\nu^{-1}}{d\nu}$ .

Specializing now to the setting where our groupoid  $G$  is the Deaconu-Renault groupoid associated to the local homeomorphism  $T$  on the compact Hausdorff space  $X$ , it is not difficult to see that the measure  $\mu$  is quasi-invariant in the fashion just described if and only if  $\mu \circ T^{-1} \ll \mu$ . In this event, if we let  $D$  denote the Radon-Nikodym derivative  $\frac{d\mu \circ T^{-1}}{d\mu}$ , then the modular function  $\Delta$  is given by the equation

$$\Delta(x, m - n, y) = \frac{D(x)D(Tx) \cdots D(T^{m-1}x)}{D(y)D(Ty) \cdots D(T^{n-1}y)}.$$

---

<sup>2</sup>If  $E$  is a  $C^*$ -correspondence over a  $C^*$ -algebra  $A$  and if  $K$  is an ideal in  $J$ , then the *relative Cuntz-Pimsner algebra* determined by  $K$ ,  $\mathcal{O}(K; E)$ , is the universal  $C^*$ -algebra for representations of  $E$ ,  $(\pi, \psi)$ , that have all the properties of a Cuntz-Pimsner representation except that the equation  $\pi(a) = (\psi, \pi)^{(1)}(\phi(a))$  is assumed to hold only for  $a \in K$ . See [28] and [18], where the basic theory of such algebras is developed.

A measurable function  $D$  defines also a transfer operator  $\mathcal{L}_D^* : M(X) \rightarrow M(X)$  by the equation

$$(9) \quad \mathcal{L}_D^*(\mu)(f) := \int_X \sum_{Ty=x} D(y)f(y) d\mu(x).$$

The relevance of the transfer operator  $\mathcal{L}_D^*$  to our situation was established by Renault in [36, Theorem 7.1] and [35, Proposition 4.2]. We state a slightly modified version of his results.

**THEOREM 5.1.** *(Renault) Let  $\mu$  be a probability measure on  $X$ . Then  $\mu$  is quasi-invariant with respect to  $G$  and admits  $\Delta$  as Radon-Nikodym derivative if and only if  $\mathcal{L}_D^*(\mu) = \mu$ .*

In applications to wavelets, i.e. to the settings where  $X = \mathbb{T}$  or  $X = \mathbb{T}^n$  and  $T$  is the power function  $z \rightarrow z^N$  or  $x + \mathbb{Z}^n \rightarrow Ax + \mathbb{Z}^n$  the measure that one usually chooses is Lebesgue measure. Also, the bundle one chooses is the trivial line bundle  $\mathcal{H} = \mathbb{T} \times \mathbb{C}$  or  $\mathcal{H} = \mathbb{T}^n \times \mathbb{C}$  and the representation is the translation representation:  $U(\gamma) : \{s(\gamma)\} \times \mathbb{C} \rightarrow \{r(\gamma)\} \times \mathbb{C}$ ,

$$U(\gamma)(s(\gamma), c) = (r(\gamma), c),$$

$\gamma \in G$ . But we note that some of the recent work of Dutkay and Roysland [15, 16] can be formulated in the setting we are describing by taking more complicated bundles and representations.

## 6. An Example: Classical Wavelets

We discuss how the constructs we have described can enter into analysis of classical wavelets. In this setting, as we have indicated,  $X$  is the circle or 1-torus  $\mathbb{T}$ , the local homeomorphism  $T$  is given by squaring:  $Tz = z^2$ , the quasi-invariant measure  $\mu$  is Lebesgue measure on  $\mathbb{T}$ , the bundle  $\mathcal{H}$  is the trivial one-dimensional bundle, and the representation  $U$  is translation. The  $L^2$ -sections of  $\mathcal{H}$  is just  $L^2(\mu)$  and if  $\pi$  is the integrated form of the representation associated to this data, then  $\pi$  represents  $C(X)$  (viewed as  $\iota(C(X))$  in  $C^*(G)$ ) as multiplication operators on  $L^2(\mu)$ . Further, if  $\{m_i\}_{i=1,2}$  is a filter bank and if  $S_1$  and  $S_2$  are the isometries it determines as in Theorem 4.3, then  $\pi(S_i)\xi(z) = m_i(z)\xi(z^2)$ ,  $i = 1, 2$ . Thus  $(\pi(S_1), \pi(S_2))$  is a Cuntz family on  $L^2(\mu)$  such that

$$\begin{aligned} \pi(\iota(\alpha(f))) &= \pi(S_1)\pi(\iota(f))\pi(S_1)^* \\ &+ \pi(S_2)\pi(\iota(f))\pi(S_2)^*, \end{aligned}$$

for all  $f \in C(\mathbb{T})$ , which is equation (8). We note in passing that there are many filter banks and that given any  $m_1$  in  $\mathcal{X}$  such that  $\langle m_1, m_1 \rangle = 1$ , we obtain a filter bank  $\{m_1, m_2\}$  if we take  $m_2$  to be the function defined by the equation

$$m_2(z) := \overline{z m_1(-z)}.$$

All other possibilities for  $m_2$  are obtained from this choice by multiplying it by  $\theta(z^2)$ , where  $\theta$  is a continuous function of modulus 1.

The key to building wavelets from the Cuntz relations is to build the minimal unitary extension of  $\pi(S_1)$ . This was observed by Bratteli and Jorgensen in [2].

However, we follow Larsen and Raeburn [25] who use the inductive limit approach advanced by Douglas [9]. Here is the basic setup: Form the inductive system

$$H_n \xrightarrow{S_{m,n}} H_m$$

where  $H_n = L^2(\mathbb{T})$  for every  $n$  and the “linking maps”  $S_{m,n} : H_n \rightarrow H_m$ , are simply the powers of  $\pi(S_1)$ :  $S_{m,n} = \pi(S_1)^{m-n}$ . We let  $H_\infty$  denote the inductive limit  $\varinjlim(\{H_n\}, \{S_{m,n}\})$ , and we let  $S_{\infty,n} : H_n \rightarrow H_\infty$  denote the limit embeddings. Then there is a unique unitary  $U$  on  $H_\infty$  so that

$$US_{\infty,n+1} = S_{\infty,n+1}\pi(S_1) = S_{\infty,n},$$

for all  $n$ . This map  $U$  is the minimal isometric extension of  $\pi(S_1)$ . We want to uncover a bit more structure in  $U$ .

To this end, observe that for  $m, n \geq 0$ ,

$$\begin{aligned} (S_{m,n}\xi)(z) &= 2^{(m-n)/2}m_1(z)m_1(z^2) \\ &\dots m_1(z^{2^{(m-n)-1}})\xi(z^{2^{m-n}}). \end{aligned}$$

Analysis of Dutkay and Jorgensen in [10] and [13, Proposition 2.2] leads to an explicit identification of  $H_\infty$  with  $L^2(\mathbb{T}_\infty, \tilde{\mu})$ , where  $\mathbb{T}_\infty$  is the 2-adic solenoid and  $\tilde{\mu}$  is a measure built from Lebesgue measure on  $\mathbb{T}$  and the transfer operator associated with  $|m_1|^2$ :

$$\mathcal{L}_{m_1}(f)(z) = \frac{1}{|T^{-1}(z)|} \sum_{w^2=z} |m_1(w)|^2 f(w).$$

The representation  $\pi$  of  $C(\mathbb{T})$  on  $L^2(\mu)$  extends to a representation  $\rho$  of  $C(\mathbb{T})$  on  $L^2(\mathbb{T}_\infty, \tilde{\mu})$  via the formula

$$\rho(f)\xi(z_1, z_2, \dots) = f(z_1)\xi(z_1, z_2, \dots),$$

where  $\xi \in L^2(\mathbb{T}_\infty, \tilde{\mu})$ , and where, recall, points in  $\mathbb{T}_\infty$  are sequences  $(z_1, z_2, \dots)$  such that  $z_k = z_{k+1}^2$ , for all  $k \geq 1$ . Also, the measure  $\tilde{\mu}$  is quasi-invariant for the extension  $T_\infty$  of the map  $T$  on  $\mathbb{T}$ , defined on  $\mathbb{T}_\infty$  by the formula  $T_\infty(z_1, z_2, \dots) = (z_1^2, z_1, z_2, \dots)$ , and  $U$  is given by the formula

$$U\xi(z_1, z_2, \dots) = \xi(z_1^2, z_1, z_2, \dots)J^{\frac{1}{2}}(z_1, z_2, \dots),$$

where  $J$  is the Radon-Nikodym derivative with respect to  $\tilde{\mu}$  of the translate of  $\tilde{\mu}$  by  $T_\infty$ . The pair  $(\rho, U)$  is a covariant pair:

$$\rho(f \circ T_\infty) = U\rho(f)U^{-1},$$

for all  $f \in C(\mathbb{T})$ . To build the wavelet associated with the filter bank, we need to get from  $L^2(\mathbb{T}_\infty, \tilde{\mu})$  to  $L^2(\mathbb{R})$  in a unitary fashion that transforms  $U$  into  $D$ , which recall is given by the formula  $D\xi(x) = \sqrt{2}\xi(2x)$ , and transforms  $\rho$  into the representation  $\tilde{\rho}(f)\xi(x) = f(e^{2\pi ix})\xi(x)$ . This is accomplished with the aid of a famous theorem of Mallat.

**THEOREM 6.1.** (Mallat [26, Theorem 2]) *Suppose  $m_1$ , which is a unit vector in the  $C^*$ -correspondence  $\mathcal{X}$ , satisfies the additional two hypotheses:*

- (1) *The Fourier coefficients of  $m_1$  are  $O((1+k^2)^{-1})$ .*
- (2)  *$|m_1(1)| = 1$*
- (3) *For all  $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ ,  $m_1(e^{ix}) \neq 0$ .*

Then the product  $\prod_{k=1}^{\infty} m_1(e^{2\pi i 2^{-k}t})$  converges on  $\mathbb{R}$  and the limit,  $\phi$ , lies in  $L^2(\mathbb{R})$ . Further, for all  $x \in \mathbb{R}$

- (1)  $\phi(2x) = m_1(e^{2\pi i x})\phi(x)$
- (2)  $\sum_{k \in \mathbb{Z}} |\phi(x+k)|^2 = 1$ .

REMARK 6.2. Mallat’s hypotheses are labeled as equations (38)–(41) on page 76 of [26]. Equation (38) is our hypothesis 1, equation (39) is our hypothesis 2., and his equation (41) is our hypothesis 3. Equation (40) is the assertion that  $m_1$  is a unit vector in  $\mathcal{X}$ . We note, however, that there is a lot of “wiggle room” in these hypotheses and a lot of work has been devoted to finding their exact limits. In [5, Chapter 6], for example, Daubechies discusses aspects of this matter at length and exposes, in particular, works of Cohen and Lawton which give necessary and sufficient conditions for a unit vector  $m_1$  to be a *trigonometric polynomial* and generate a wavelet. The point to keep in mind, for our purposes, is that a unit vector  $m_1 \in \mathcal{X}$  always generates an isometry  $S_1$ . Further, the minimal unitary extension of  $\pi(S_1)$ ,  $U$ , lives on the space  $L^2(\mathbb{T}_\infty, \tilde{\mu})$ , where  $\tilde{\mu}$  is constructed using  $m_1$ . These things do not depend on anything other than the fact that  $m_1$  is a unit vector in  $\mathcal{X}$ . However, *some* hypotheses on  $m_1$  seem to be necessary to get from  $L^2(\mathbb{T}_\infty, \tilde{\mu})$  to  $L^2(\mathbb{R})$ . Conclusion 1. of Mallat’s theorem is the stepping stone that takes us from  $L^2(\mathbb{T}_\infty, \tilde{\mu})$  to  $L^2(\mathbb{R})$ . Conclusion 2. does not play an immediate role in the Larsen-Raeburn approach, but it implies, in particular, that translates of  $\phi$  are orthonormal [26, Equation (50)].

With the aid of  $\phi$  we may define  $R_n : H_n \rightarrow L^2(\mathbb{R})$  via the formula  $(R_n \xi)(x) := 2^{-\frac{n}{2}} \xi(e^{2\pi i(2^{-n}x)})\phi(2^{-n}x)$ ,  $\xi \in H_n (= L^2(\mathbb{T}))$ . It then is a simple matter to check that  $R_n$  is an isometry that satisfies the equation  $R_{n+1}\pi(S_1) = R_n$ . By properties of inductive limits, we may conclude that there is a unique Hilbert space isometric injection  $R_\infty : H_\infty \rightarrow L^2(\mathbb{R})$  so that  $R_\infty S_{\infty,n} = R_n$ . The problem now is to show that  $R_\infty$  is surjective. For this purpose, define  $\mathcal{V}_n := R_n H_n = R_n L^2(\mathbb{T})$ . Then, as Mallat showed in the proof of the second half of Theorem 2 in [26], we have:

LEMMA 6.3. (*Mallat*)

- (1)  $\mathcal{V}_n \subseteq \mathcal{V}_{n+1}$ .
- (2)  $\bigcap \mathcal{V}_n = \{0\}$ .
- (3)  $\bigvee \mathcal{V}_n = L^2(\mathbb{R})$ .

Thus  $R_\infty : H_\infty \rightarrow L^2(\mathbb{R})$  is a Hilbert space isomorphism.

We want to remark that conditions 2. and 3. of the preceding lemma represent two different problems. Condition 2. is the assertion that the isometry  $\pi(S_1)$  is a pure isometry. Bratteli and Jorgensen provide a proof of this that is different from Mallat’s by noting that  $\pi(S_1)$  is pure because  $m_1$  does not have modulus one a.e. [2, Theorem 3.1]. The condition 3. also has alternate proofs. One that we find particularly attractive, because it works in the more general setting of wavelets built on  $\mathbb{T}^n$  using a dilation matrix  $A$ , may be found in Strichartz’s survey [38, Lemma 3.1].

Recall that the “dilation by 2 operator”,  $D$ , is defined on  $L^2(\mathbb{R})$  by the formula  $D\xi(x) = 2^{1/2}\xi(2x)$  and that  $D$  a unitary operator on  $L^2(\mathbb{R})$ .

LEMMA 6.4.  $R_\infty U R_\infty^{-1} = D$ , and  $R_\infty \rho(f) R_\infty^{-1} \xi(t) = f(e^{2\pi i t})\xi(t)$  for all  $f \in C(\mathbb{T})$ , all  $\xi$  in  $L^2(\mathbb{R})$  and all  $t \in \mathbb{R}$ .

PROOF. The first assertion requires only a simple calculation:

$$\begin{aligned} D(R_\infty(S_{\infty,n+1}\xi))(x) &= D(R_{n+1}\xi)(x) = 2^{1/2}(R_{n+1}\xi)(2x) \\ &= 2^{1/2}2^{-\frac{(n+1)}{2}}\xi(e^{2\pi i(2^{-(n+1)}2x)})\phi(2^{-(n+1)}2x) = (R_n\xi)(x) \\ &= (R_\infty S_{\infty,n}\xi)(x) = (R_\infty U S_{\infty,n+1}\xi)(x). \end{aligned}$$

The second assertion is verified similarly. □

If we set  $\mathcal{W}_n = \mathcal{V}_{n+1} \ominus \mathcal{V}_n$ , then by Lemma 6.3:

$$\bigoplus_{n \in \mathbb{Z}} \mathcal{W}_n = L^2(\mathbb{R}).$$

But then we find that

$$\begin{aligned} (10) \quad \mathcal{W}_0 &= \mathcal{V}_1 \ominus \mathcal{V}_0 = R_1 L^2(\mathbb{T}) \ominus R_0 L^2(\mathbb{T}) \\ &= R_1 L^2(\mathbb{T}) \ominus R_1 \pi(S_1) L^2(\mathbb{T}) \\ &= R_1(L^2(\mathbb{T}) \ominus \pi(S_1)L^2(\mathbb{T})) \\ &= R_1 \pi(S_2) L^2(\mathbb{T}), \end{aligned}$$

by the Cuntz relations. So if we set  $e_k(z) = z^k$  and set  $\zeta_k := R_1 \pi(S_2)e_{-k}$ . Then  $\{\zeta_k\}$  is an orthonormal basis for  $\mathcal{W}_0$  and

$$\begin{aligned} \zeta_k(x) &= 2^{-1/2}(2^{1/2}m_2(e^{(2\pi i2^{-1}x)}))e^{-2\pi ikx}\phi(2^{-1}x) \\ &= e^{-2\pi ikx}\zeta(x) \end{aligned}$$

where

$$\zeta(x) := m_2(e^{\pi ix})\phi(2^{-1}x).$$

Thus,  $\{\zeta_{j,k}\}_{j,k=-\infty}^\infty$  is an orthonormal basis for  $L^2(\mathbb{R})$ , where

$$\begin{aligned} \zeta_{j,k}(x) &= D^j \zeta_k(x) \\ &= 2^{j/2}e^{(-2\pi ik2^j x)}\zeta(2^j x). \end{aligned}$$

Consequently, if  $\psi$  is the inverse Fourier transform of  $\zeta$ , then  $\psi$  is a wavelet.

This completes the proof of the following theorem as formulated by Bratteli and Jorgensen [2] and Larsen and Raeburn [25].

**THEOREM 6.5.** (*Bratteli-Jorgensen, Larsen-Raeburn*)  
*The inverse Fourier transform of*

$$m_2(e^{\pi ix})\phi(2^{-1}x)$$

*is the wavelet associated with the filter bank  $(m_1, m_2)$ .*

### 7. Further Thoughts: Fractafolds

As we have seen, the  $C^*$ -algebra  $C^*(G)$  always contains an isometry  $S$  and a Cuntz family of isometries  $\{S_i\}_{i=1}^n$ , provided  $\mathcal{X}$  has an orthonormal basis. Further, we may construct the minimal unitary extension of either  $S$  or of any of the  $S_i$  *essentially within*  $C^*(G)$ . More accurately, these objects are constructed in the multiplier algebra of the  $C^*$ -algebra of a Morita equivalent groupoid that we denote by  $G_\infty$ . To construct  $G_\infty$ , we form an analogue of the 2-adic solenoid, viz., the projective limit space  $X_\infty := \{\underline{x} := (x_1, x_2, \dots) : T(x_{k+1}) = x_k\}$ , and we set

$$G_\infty = \{(\underline{x}, n - m, \underline{y}) : T^n x_1 = T^m y_1\}.$$

Then  $G_\infty$  is a groupoid with unit space  $X_\infty$  that is Morita equivalent to  $G$  in the sense of [27]. Note, however, that  $G_\infty$  is not  $r$ -discrete. It has many Haar systems that are not in any evident way equivalent. Each transfer operator  $\mathcal{L}_D^*$  associated with a continuous function  $D$  as in equation (9) determines a natural Haar system on  $G_\infty$  that reflects special features of  $C^*(G)$ . In particular, if  $D = |m|$  where  $m$  is a unit vector in  $\mathcal{X}$  (so in particular if  $m$  is part of an orthonormal basis for  $\mathcal{X}$ ), then the minimal unitary extension of  $S_1 = \iota(m)S$  (in the notation of Theorem 3.3) lives in the multiplier algebra of  $C^*(G_\infty)$ , when  $G_\infty$  is endowed with the Haar system determined by  $|m|$ .

We observe in passing that Dutkay and Jorgensen [13] associate the  $C^*$ -crossed product  $C(X_\infty) \rtimes \mathbb{Z}$  to the setting we have been discussing (where  $\mathbb{Z}$  is viewed as acting on  $X_\infty$  through the homeomorphism  $T_\infty$ , which is defined via the formula  $T_\infty(x_1, x_2, \dots) = (Tx_1, x_1, x_2, \dots)$ ). This crossed product lies in the multiplier algebra of  $C^*(G_\infty)$ .

Thus, our analysis shows that the study of wavelets can be broken into two pieces. First, there are the structures that are intrinsic to the geometric setting of a space  $X$  with a local homeomorphism  $T$ . These include the groupoid  $G$  and its  $C^*$ -algebra, the pseudogroup  $\mathfrak{G}$ , and the Deaconu correspondence  $\mathcal{X}$ . These are the source of isometries and the Cuntz relations - assuming  $\mathcal{X}$  has an orthonormal basis. Each choice of orthonormal basis gives Cuntz isometries in  $C^*(G)$  that satisfy equation (8). Even if  $\mathcal{X}$  fails to have an orthonormal basis,  $\mathcal{X}$  will always contain a (*normalized tight*) *frame* in the sense of Frank and Larson [19, Definition 3.1] (also called a *quasi-basis* in the sense of Watatani [40]). This is a collection of vectors  $\{\psi_i\}_{i=1}^n$  such that for every  $\xi \in \mathcal{X}$ ,  $\xi = \sum_{i=1}^n \psi_i \langle \psi_i, \xi \rangle$  and  $\langle \xi, \xi \rangle = \sum_{i=1}^n \langle \xi, \psi_i \rangle \langle \psi_i, \xi \rangle$ . Such a collection may be constructed easily with the aid of a partition of unity subordinate to an open cover of  $X$  such that  $T$  is a homeomorphism when restricted to each element of the cover. Much of the analysis in  $C^*(G)$  can be accomplished with a frame for  $\mathcal{X}$ . The parameters involved in representing the Cuntz relations on Hilbert space come from the representation theory of  $C^*(G)$ . Even constructing the minimal unitary extension of  $\pi(S_1)$  involves ingredients intrinsic to our setting. The groupoid  $G_\infty$  is Morita equivalent to  $G$  and carries a natural Haar system that may be “pegged” to  $S_1$  - more accurately, a natural Haar system on  $G_\infty$  can be constructed from each low pass filter. There will result a natural multiresolution analysis in  $L^2(X_\infty, \tilde{\mu})$ .

To make contact with wavelet basis in  $L^2(\mathbb{R}^n)$  for some  $n$ , which is the second piece in the study of wavelets, one must have a mechanism for passing from  $L^2(X_\infty, \tilde{\mu})$  to  $L^2(\mathbb{R}^n)$ . This involves a different set of tools. In the final analysis, there may not be any naturally constructed wavelet-like bases in  $L^2(\mathbb{R}^n)$  coming from a particular space and local homeomorphism. One should not despair at this. Rather, one should focus on building orthonormal bases in  $W_0$  (the wandering subspace in equation (10) and then push them around to form an orthonormal basis for all of  $L^2(X_\infty, \tilde{\mu})$  using the minimal unitary extension  $U$  of  $\pi(S_1)$ . After all,  $L^2(X_\infty, \tilde{\mu})$  and the other spaces we have been discussing are the *naturally* occurring spaces adapted to  $X$  and  $T$ . This effectively is what Dutkay and Jorgensen did in [11] and is similar to what Jorgensen and Pedersen did in [22].

We believe the proof of Theorem 6.5 that we presented, which is due to Larsen and Raeburn [25], can be tweaked to show a bit more. The 2-adic solenoid  $\mathbb{T}_\infty$  is the dual group of the 2-adic numbers: the set of all rational numbers whose

denominators are powers of 2, positive and negative. Since the 2-adic numbers form a dense subgroup of  $\mathbb{R}$ ,  $\mathbb{T}_\infty$  contains a dense copy of  $\mathbb{R}$ . We believe the measure  $\tilde{\mu}$  is supported on this copy of  $\mathbb{R}$  and is mutually absolutely continuous with respect to Lebesgue measure transported there. The mapping  $R_\infty$  ought to be, then, just multiplication by (the square root of) a suitable Radon-Nikodym derivative.

In another direction, which we find very piquant, we can find Cuntz families in the  $C^*$ -algebras or their multiplier algebras of other groupoids that are Morita equivalent to  $G$ . This raises the prospect of carrying out groupoid-like harmonic analysis using Cuntz families of isometries on other spaces that Strichartz has called *fractafolds* - i.e., spaces that are locally like fractals [39]. The point is that under favorable circumstances  $G$  is the groupoid of germs of the pseudogroup  $\mathfrak{G}$  of partial homeomorphisms defined by  $T$ . We believe the pseudogroup of partial homeomorphisms of a fractafold that is locally like  $X$  will be Morita equivalent, in a sense described by Renault in [34, Section 3], to  $\mathfrak{G}$ . This sense is based on work of Kumjian [23] and Haefliger [20]. At this stage, however, there still is a lot of work to do to substantiate this belief.

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