MAULDIN-WILLIAMS GRAPHS, MORITA EQUIVALENCE
AND ISOMORPHISMS

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ABSTRACT. We describe a method for associating some non-self-adjoint algebras to Mauldin-Williams graphs and we study the Morita equivalence and isomorphism of these algebras.

We also investigate the relationship between the Morita equivalence and isomorphism class of the $C^\ast$-correspondences associated with Mauldin-Williams graphs and the dynamical properties of the Mauldin-Williams graphs.

1. Introduction

In this note we follow the notation from [8]. By a Mauldin-Williams graph (see [14]), we mean a system $G = (G, \{T_v, \rho_v\}_{v \in V}, \{\phi_e\}_{e \in E})$, where $G = (V, E, r, s)$ is a graph with a finite set of vertices $V$, a finite set of edges $E$, a range map $r$ and a source map $s$, and where $\{T_v, \rho_v\}_{v \in V}$ and $\{\phi_e\}_{e \in E}$ are families such that:

1. Each $T_v$ is a compact metric space with a prescribed metric $\rho_v, v \in V$.
2. For $e \in E$, $\phi_e$ is a continuous map from $T_r(e)$ to $T_s(e)$ such that
   
   \[ c_1 \rho_r(e)(x, y) \leq \rho_s(e)(\phi_e(x), \phi_e(y)) \leq c_2 \rho_s(e)(x, y) \]

   for some constants $c_1, c_2$ satisfying $0 < c_1 \leq c_2 < 1$ (independent of $e$) and all $x, y \in T_r(e)$.

We shall assume, too, that the source map $s$ and the range map $r$ are surjective. Thus, we assume that there are no sinks and no sources in the graph $G$.

In [8] we associated to a Mauldin-Williams graph $G = (G, \{T_v, \rho_v\}_{v \in V}, \{\phi_e\}_{e \in E})$ a so-called $C^\ast$-correspondence $X$ over the $C^\ast$-algebra $A = C(T)$, where $T = \bigsqcup_{v \in V} T_v$ is the disjoint union of the spaces $T_v, v \in V$, as follows. Let $E \times_G T = \{(e, x) | x \in T_r(e)\}$. Then, by our finiteness assumptions, $E \times_G T$ is a compact space. We set $X = C(E \times_G T)$ and view $X$ as a $C^\ast$-correspondence over $C(T)$ via the formulae:

\[ \xi \cdot a(e, x) := \xi(e, x)a(x), \]
\[ a \cdot \xi(e, x) := a \circ \phi_e(x)\xi(e, x) \]

and

\[ \langle \xi, \eta \rangle_A(x) := \sum_{e \in E, x \in T_r(e)} \overline{\xi(e, x)}\eta(e, x), \]

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where \( a \in C(T) \) and \( \xi, \eta \in C(E \times_G T) \). With these data we can form the tensor algebra \( T_+(\mathcal{X}) \) as prescribed in [15] and [16]. Our main result is:

**Theorem 1.1.** For \( i = 1, 2 \), let \( \mathcal{G}_i = (G_i, (K^i_v)_{v \in V}, (\phi^i_e)_{e \in E_i}) \) be two Mauldin-Williams graphs. Let \( A_i = C(K^i) \) and let \( \mathcal{X}_i \) be the associated \( C^* \)-algebras and \( C^* \)-correspondences. Then the following are equivalent:

1. \( T_+(\mathcal{X}_1) \) is strongly Morita equivalent to \( T_+(\mathcal{X}_2) \) in the sense of [2].
2. \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are strongly Morita equivalent in the sense of [10].
3. \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \) are isomorphic as \( C^* \)-correspondences.
4. \( T_+(\mathcal{X}_1) \) is completely isometrically isomorphic to \( T_+(\mathcal{X}_2) \).

We find this result especially remarkable in light of Theorem 2.3 from [8] Theorem 1.1] (see also Section 4.2 from [18]), which states that the Cuntz-Pimsner algebra of the Mauldin-Williams graph, we study the isomorphism class of our graphs, \( (G, \{T_v, \rho_v\}_{v \in V}, \{\phi_e\}_{e \in E}) \), is isomorphic, but have the same \( C^* \)-envelope, namely \( \mathcal{O}_n \).

To understand further the relationship between the tensor algebra and the Mauldin-Williams graph, we study the isomorphism class of our \( C^* \)-correspondences and tensor algebras in terms of the dynamics of the Mauldin-Williams graph. Roughly, we find that two \( C^* \)-correspondences associated to two Mauldin-Williams graphs, \( (G_i, (K^{i}_v)_{v \in V}, (\phi^i_e)_{e \in E_i}) \), \( i = 1, 2 \), are isomorphic if the maps \( \phi^1_e \) and \( \phi^2_e \) are locally conjugate in a sense that will be made precise later.

2. **Non-self-adjoint algebras associated with Mauldin-Williams graphs**

**Definition 2.1.** An invariant list associated with a Mauldin-Williams graph \( \mathcal{G} = (G, \{T_v, \rho_v\}_{v \in V}, \{\phi_e\}_{e \in E}) \) is a family \( (K_v)_{v \in V} \) of compact sets, such that \( K_v \subset T_v \), for all \( v \in V \) and such that

\[
K_v = \bigcup_{e \in E, \phi(e) = v} \phi_e(K_{\phi(e)}).
\]

Since each \( \phi_e \) is a proper contraction, \( \mathcal{G} \) has a unique invariant list (see [13] Theorem 1). We set \( T := \bigcup_{v \in V} T_v \) and \( K := \bigcup_{v \in V} K_v \) and we call \( K \) the invariant set of the Mauldin-Williams graph.

In the particular case when we have one vertex \( v \) and \( n \) edges, i.e. in the setting of an iterated function system, the invariant set is the unique compact subset \( K := K_v \) of \( T = T_v \) such that

\[
K = \phi_1(K) \cup \cdots \cup \phi_n(K).
\]

Note that the \( * \)-homomorphism \( \Phi : A \to \mathcal{L}(\mathcal{X}) \), \( (\Phi(a)\xi)(e,x) = a \circ \phi_e(x)\xi(e,x) \), which gives the left action of the \( C^* \)-correspondence associated to a Mauldin-Williams graph, is faithful if and only if \( K = T \). In this note we assume that \( T \) equals the invariant set \( K \).

Kajiwara and Watatani have proved in [10], Lemma 2.3] that, if the contractions are proper, the invariant set of an iterated function system has no isolated point. Their proof can be easily generalized to the invariant set of a Mauldin-Williams graph. Hence \( K \) has no isolated points.

For a \( C^* \)-correspondence \( \mathcal{X} \) over a \( C^* \)-algebra \( A \), the (full) Fock space over \( \mathcal{X} \) is

\[
\mathcal{F}(\mathcal{X}) = A \oplus \mathcal{X} \oplus \mathcal{X} \otimes 2 \oplus \cdots.
\]
We write \( \Phi_\infty \) for the left action of \( A \) on \( \mathcal{F}(\mathcal{X}) \), \( \Phi_\infty(a) = \text{diag}(a, \Phi^{(1)}(a), \Phi^{(2)}(a), \cdots) \), where \( \Phi^{(n)} \) is the left action of \( A \) on \( \mathcal{X}^{\otimes n} \) (\( \Phi^{(1)} = \Phi \), the left action of \( A \) on \( \mathcal{X} \)). For \( \xi \in \mathcal{X} \), the creation operator determined by \( \xi \) is defined by the formula \( T_\xi(\eta) = \xi \otimes \eta \), for all \( \eta \in \mathcal{F}(\mathcal{X}) \).

**Definition 2.2.** The tensor algebra of \( \mathcal{X} \), denoted by \( T_+\mathcal{X} \), is the norm closed subalgebra of \( \mathcal{L}(\mathcal{F}(\mathcal{X})) \) generated by \( \Phi_\infty(A) \) and the creation operators \( T_\xi \), for \( \xi \in \mathcal{X} \) (see [15, Lemma 5.2]). The \( C^* \)-algebra generated by \( T_+\mathcal{X} \) is denoted by \( T(\mathcal{X}) \) and it is called the Toeplitz algebra of the \( C^* \)-correspondence \( \mathcal{X} \).

We may regard each finite sum \( \sum_{n=0}^N \mathcal{X}^{\otimes n} \) as a subspace of \( \mathcal{F}(\mathcal{X}) \) and we may regard \( \mathcal{L}(\sum_{n=0}^N \mathcal{X}^{\otimes n}) \) as a subalgebra of \( \mathcal{L}(\mathcal{F}(\mathcal{X})) \) in the obvious way. Let \( B \) be the \( C^* \)-subalgebra of \( \mathcal{L}(\mathcal{F}(\mathcal{X})) \) generated by all the \( \mathcal{L}(\sum_{n=0}^N \mathcal{X}^{\otimes n}) \) as \( N \) ranges over the non-negative integers. Then \( T(\mathcal{X}) \subset M(B) \), the multiplier algebra of \( B \). The Cuntz-Pimsner algebra \( \mathcal{O}(\mathcal{X}) \) is defined to be the image of \( T(\mathcal{X}) \) in the corona algebra \( M(B)/B \) (see [15] and [17]).

By a homomorphism from an \( A_1 - B_1 \) \( C^* \)-correspondence \( \mathcal{X}_1 \), to an \( A_2 - B_2 \) \( C^* \)-correspondence \( \mathcal{X}_2 \) we mean a triple \((\alpha, V, \beta)\), where \( \alpha : A_1 \to A_2, \beta : B_1 \to B_2 \) are \( C^* \)-homomorphisms and \( V : \mathcal{X}_1 \to \mathcal{X}_2 \) is a linear map such that \( V(\alpha a b) = \alpha(a) V(\xi) \beta(b) \) and such that \( \langle V(\xi), V(\eta) \rangle_{B_2} = \beta(\langle \xi, \eta \rangle_{B_1}) \) (see [16] Section 1). When \( A_1 = A_2 \) and \( B_1 = B_2 \), we will consider \( \alpha \in \text{Aut}(A_1) \) and \( \beta \in \text{Aut}(B_1) \). This then forces \( V \) to be isometric. If \( V \) is also surjective, we shall say that \( V \) is a correspondence isomorphism over \((\alpha, \beta)\). If, moreover, \( A_1 = B_1 \) and \( \alpha = \beta \), we say that \( V \) is a correspondence isomorphism over \( \alpha \).

A central concept for our work in this note is the strong Morita equivalence for \( C^* \)-correspondences defined in [16] Definition 2.1], which we review here.

**Definition 2.3.** If \( \mathcal{X} \) is a \( C^* \)-correspondence over a \( C^* \)-algebra \( A \), and \( \mathcal{Y} \) is a \( C^* \)-correspondence over a \( C^* \)-algebra \( B \), we say that \( \mathcal{X} \) and \( \mathcal{Y} \) are strongly Morita equivalent if \( A \) and \( B \) are strongly Morita equivalent via an \( A \)-\( B \)-bi-module \( \mathcal{Z} \) (in which case we write \( A \ \text{SME} \sim \ Z \ B \)), for which there is an \( A \)-\( B \)-correspondence isomorphism \( (id, W, id) \) from \( \mathcal{Z} \otimes_B \mathcal{Y} \) onto \( \mathcal{X} \otimes_A \mathcal{Z} \). This means, in particular, that \( W(a \xi b) = a W(\xi) b \) for all \( a \in A, b \in B \) and \( \xi \in \mathcal{Z} \otimes_B \mathcal{Y} \) and that \( \langle W(\xi), W(\eta) \rangle_B = \langle \xi, \eta \rangle_B \).

We say that a \( C^* \)-correspondence \( \mathcal{X} \) over a \( C^* \)-algebra \( A \) is aperiodic if: for all \( n \geq 1 \), for all \( \xi \in \mathcal{X}^{\otimes n} \) and for all hereditary subalgebras \( B \subseteq A \), we have

\[
\inf \left\{ \| \Phi^{(n)}(a) \xi a \| \mid a \geq 0, a \in B, \| a \| = 1 \right\} = 0.
\]

It was proved in [16] Theorem 3.2, Theorem 3.5] that if \( \mathcal{X} \) and \( \mathcal{Y} \) are strongly Morita equivalent, then \( T_+\mathcal{X} \) and \( T_+\mathcal{Y} \) (respectively \( T(\mathcal{X}) \) and \( T(\mathcal{Y}) \), \( \mathcal{O}(\mathcal{X}) \) and \( \mathcal{O}(\mathcal{Y}) \)) are strongly Morita equivalent. Also, if \( \mathcal{X} \) and \( \mathcal{Y} \) are aperiodic \( C^* \)-correspondences over the \( C^* \)-algebras \( A \) and \( B \), respectively, and if \( T_+\mathcal{X} \) and \( T_+\mathcal{Y} \) are strongly Morita equivalent in the sense of [2], then \( \mathcal{X} \) and \( \mathcal{Y} \) are strongly Morita equivalent (see [16] Theorem 7.2).)

To study the aperiodicity and strong Morita equivalence of \( C^* \)-correspondences associated to Mauldin-Williams graphs, we need the following lemma which gives an equivalent description of when a \( C^* \)-correspondence is aperiodic.

**Lemma 2.4** ([16] Lemma 5.2]. The \( C^* \)-correspondence \( \mathcal{X} \) is aperiodic if and only if given \( a_0 \in A, a_0 \geq 0, \xi^k \in \mathcal{X}^{\otimes k}, 1 \leq k \leq n, \) and \( \varepsilon > 0 \), there is an \( x \) in the
hereditary subalgebra $\overline{a_0 A a_0}$, with $x \geq 0$ and $\|x\| = 1$, such that

$$\|x a_0 x\| > \|a_0\| - \varepsilon$$

and

$$\|\Phi^{(k)}(x)\xi^k x\| < \varepsilon \text{ for } 1 \leq k \leq n.$$ 

For a directed graph $G = (V, E, r, s)$ and for $k \geq 2$, we define

$$E^k := \{\alpha = (\alpha_1, \ldots, \alpha_k) : \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}), i = 1, \ldots, k-1\}$$

to be the set of paths of length $k$ in the graph $G$. We define also the infinite path space to be

$$E^\infty := \{(\alpha_i)_{i \in \mathbb{N}} : \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \text{ for all } i \in \mathbb{N}\}.$$ 

For $\alpha \in E^k$, we write $\phi_\alpha = \phi_{\alpha_1} \circ \cdots \circ \phi_{\alpha_k}$. 

**Proposition 2.5.** Let $G = (G, (K_\alpha)_{\alpha \in V}, (\phi_e)_{e \in E})$ be a Mauldin-Williams graph with the invariant set $K$. Let $A = C(K)$ be the associated C*-algebra and let $X$ be the associated C*-correspondence. Then the $C^*$-correspondence $X$ is aperiodic.

**Proof.** Note that $\phi_\alpha : K_{r(\alpha)} \to K_{s(\alpha)}$, with $\alpha \in E^k$ and $k \in \mathbb{N}$, has a fixed point if and only if $r(\alpha) = s(\alpha)$, i.e. $\alpha$ is a cycle in the graph $G$.

Fix $n_0 \in \mathbb{N}$, choose $k \in \mathbb{N}, 1 \leq k \leq n_0$; let $a_0 \in A$ with $a_0 \geq 0$; let $\xi \in X^{\otimes k}$ and let $\varepsilon > 0$. We verify the criterion in Lemma 2.4 first when $n_0 = k = 1$.

Without loss of generality, we assume that $\|a_0\| = 1$. Then we can find $t_0 \in K$ such that $|a_0(t_0)| \geq 1 - \varepsilon$ and $t_0$ is not a fixed point for any $\phi_e$, $e \in E$. Let $v_0 \in V$ be such that $t_0 \in K_{v_0}$. Choose $\delta_1 > 0$ such that $B(t_0, \delta_1) \subset K_{v_0}$ and $B(\phi_e(t_0), \delta_1) \cap B(t_0, \delta_1) = \emptyset$ for all $e \in E$ for which $r(e) = v_0$. Let

$$\delta_2 := \begin{cases} \min\{\rho_{v_0}(t_0, t) \mid a_0(t) = 0\}, & \text{if } \{t \in K_{v_0} : a_0(t) = 0\} \neq \emptyset, \\ \delta_1, & \text{otherwise.} \end{cases}$$

Set $\delta = \min\{\delta_1, \delta_2\}$ and let $x \in A, x \geq 0$ be such that

$$x(t) = \begin{cases} 1, & \text{if } t = t_0, \\ 0, & \text{if } t \in K \setminus B(t_0, \delta). \end{cases}$$

Since $x(t) > 0$ only when $a_0(t) > 0$, it follows that $x \in \overline{a_0 A a_0}$. Moreover, $x(t_0) a_0(t_0) x(t_0) > 1 - \varepsilon$, hence $\|x a_0 x\| > 1 - \varepsilon$.

Fix $t \in K$. If $t \in B(t_0, \delta)$, then $\phi_e(t) \notin B(t_0, \delta)$, by our choice of $\delta_1$ and the fact that each map $\phi_e$ is a contraction, for all $e \in E$ such that $r(e) = v_0$; so $x \circ \phi_e(t) x(t) = 0$. If $t \notin B(t_0, \delta)$, then $x(t) = 0$, hence $x \circ \phi_e(t) x(t) = 0$, for all $e \in E$ such that $t \in K_{r(e)}$. Therefore, $\langle \Phi(x) \xi x, \Phi(x) \xi x \rangle_A(t) = \sum_{e \in E, t \in K_{r(e)}} (x \circ \phi_e(t))^2 |\xi(e, t)|^2 x(t)^2$, we see that $\|\Phi(x) \xi x\| = 0$.

For $n_0 = 2$, we choose $t_0 \in K$ such that $a_0(t_0) > 1 - \varepsilon$ and $t_0$ is not a fixed point for any $\phi_\alpha$ with $\alpha \in E^2$. Let $v_0 \in V$ be such that $t_0 \in K_{v_0}$. Let $\delta_1 > 0$ be such
that $B(\phi_\alpha(t_0), \delta_1) \cap B(t_0, \delta_1) = \emptyset$, for all $\alpha \in \mathbb{E}^2$ for which $r(\alpha) = \nu_0$, and such that $B(t_0, \delta_1) \subset K_{\nu_0}$. Choosing $\delta_2, \delta$ and $x$ as before, we conclude that $x \in a_0 A a_0$ and $\|x\| > 1 - \varepsilon$. Moreover, we have $x \circ \phi_\alpha(t)x = 0$ for all $t \in K$, $\alpha \in \mathbb{E} \cup \mathbb{E}^2$ (since $\phi_\alpha$ is a contraction, for all $\alpha \in \mathbb{E} \cup \mathbb{E}^2$); and since

$$\left\langle \Phi^{(2)}(x) \xi^2 x, \Phi^{(2)}(x) \xi^2 x \right\rangle_A(t) = \sum_{\alpha \in K^2 \cap K_r} \alpha(t) \|\xi_2^2(\alpha_2, t)\|^2 |\xi_1^2(\alpha_1, \phi_\alpha(t))|^2 x(t)^2 = 0,$$

it follows that $\|\Phi^{(k)}(x) \xi^k x\| = 0$ for $k = 1, 2$. Applying the same argument inductively, we see that $\mathcal{X}$ is an aperiodic $C^*$-correspondence. \qed

Let $K^1$ and $K^2$ be two compact metric spaces. Let $A_1 = C(K^1)$ and $A_2 = C(K^2)$. If $A_1 \simeq_{\text{SMIE}} A_2$, then the Rieffel correspondence determines a unique homeomorphism $f : K^1 \rightarrow K^2$ and a unique Hermitian line bundle $\mathcal{L}$ over $\text{Graph}(f) = \{(x, f(x)) : x \in K^1\}$, such that $\mathcal{Z}$ is isomorphic to $\Gamma(\mathcal{L})$ (see [21], [20] Section 3.3 and Example 4.55), [19] Appendix (A)), where $\Gamma(\mathcal{L})$ is the imprimitivity bimodule of the cross sections of $\mathcal{L}$ endowed with the following structure:

$$(a \cdot s \cdot b)(x, f(x)) = a(x)s(x, f(x))b(f(x)), \quad \langle s_1, s_2 \rangle_{A_2}(y) = s_1(f^{-1}(y), y)s_2(f^{-1}(y), y),$$

$$A_1 \langle s_1, s_2 \rangle(x) = s_1(x, f(x))s_2(x, f(x)),$$

for all $a \in A_1$, $b \in A_2$, $s, s_1, s_2 \in \Gamma(\mathcal{L})$. We write $\mathcal{Z}(f, \mathcal{L})$ for $\Gamma(\mathcal{L})$.

We are ready to prove the main theorem.

\textbf{Proof of Theorem 1.1.} By Proposition 2.5, $\mathcal{X}_1$ and $\mathcal{X}_2$ are aperiodic $C^*$-correspondences. Using [16] Theorem 7.2, we obtain that (1) implies (2).

Now we show that (2) implies (3). Suppose that $\mathcal{X}_1$ and $\mathcal{X}_2$ are strongly Morita equivalent. This implies that $A_1$ and $A_2$ are strongly Morita equivalent via an imprimitivity bimodule $\mathcal{Z}$ such that $\mathcal{Z} \otimes \mathcal{X}_2$ is isomorphic to $\mathcal{X}_1 \otimes \mathcal{Z}$. Let $f : K^1 \rightarrow K^2$ and $\mathcal{L}$ be the homeomorphism and the line bundle determined by the Rieffel correspondence. We have that $\mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2$ is isomorphic to $\mathcal{X}_1 \otimes \mathcal{Z}(f, \mathcal{L})$. Hence $\mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \mathcal{Z}(f, \mathcal{L})$ is isomorphic to $\mathcal{X}_1$, where $\mathcal{Z}(f, \mathcal{L})$ is the dual imprimitivity bimodule (see [20] Proposition 3.18). We prove that $\mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \mathcal{Z}(f, \mathcal{L})$ is isomorphic to $\mathcal{X}_2$ over an isomorphism $\alpha$ of $A_1$ and $A_2$.

Let $\alpha : A_1 \rightarrow A_2$ be defined by the formula $\alpha(a) = a \circ f^{-1}$ and let $V : \mathcal{Z}(f, \mathcal{L}) \otimes \mathcal{X}_2 \otimes \mathcal{Z}(f, \mathcal{L}) \rightarrow \mathcal{X}_2$ be defined by the formula

$$V(s_1 \otimes \xi \otimes \bar{s}_2)(e, y) = s_1(f^{-1}(\phi_\xi^2(y), \phi_\xi^2(y))) \xi(e, x)s_2(f^{-1}(y), y).$$

Then $\alpha$ is an isomorphism and

$$V(a \cdot s_1 \otimes \xi \otimes \bar{s}_2 \cdot b) = a \cdot V(s_1 \otimes \xi \otimes \bar{s}_2) \cdot b,$$
for all \(a, b \in A, s_1, s_2 \in \mathcal{Z}(f, L), \xi \in \mathcal{X}_2\). Moreover, we have that
\[
\langle V(s_1 \otimes \xi \otimes s_2), V(t_1 \otimes \eta \otimes t_2) \rangle_{A_2}(y) = \sum_{e \in E} \sum_{y \in K_{i(e)}^y} \left( s_1(f^{-1}(\phi_e^1(y))), \phi_e^2(y) \right) \xi(e, x) s_2(f^{-1}(y), y)
\]
for all \(s_1, s_2, t_1, t_2 \in \mathcal{Z}(f, L)\) and \(\xi, \eta \in \mathcal{X}_2\). Also, for \(\xi \in \mathcal{X}_2, V(1 \otimes \xi \otimes 1) = \xi\). Hence \(V\) is a correspondence isomorphism. Thus \(X_1\) is isomorphic to \(X_2\). The rest is clear.

It was shown in \[8\] Theorem 2.3 that the Cuntz-Pimsner algebra of the C*-correspondence built from a Mauldin-Williams graph is isomorphic to the Cuntz-Krieger algebra of the underlying graph \(G = (V, E, r, s)\) (as defined in \[12\]). Hence, for C*-correspondences associated to Mauldin-Williams graphs with the same underlying graph which are not isomorphic, we obtain tensor algebras which are not Morita equivalent, but have the same C*-envelope, namely the Cuntz-Krieger algebra of the graph \(G\).

3. The isomorphism class of the C*-correspondences associated with Mauldin-Williams graphs

In the following we analyze the relation between the isomorphism class of the C*-correspondences associated with two Mauldin-Williams graphs, \(G_i = (G_i, (K_i^1), (\phi_e^i)_{e \in E}, i = 1, 2, \) and the topological and dynamical properties of the Mauldin-Williams graphs.

Since, by \[18\] Section 4.2] and \[8\] Theorem 2.3, the Cuntz-Pimsner algebra associated to a Mauldin-Williams graph depends only on the structure of the underlying graph \(G\), we will consider only Mauldin-Williams graphs having the same underlying graph \(G = (V, E, r, s)\).

Next we determine necessary and sufficient conditions for the isomorphism of the C*-correspondences associated to two Mauldin-Williams graphs.

**Proposition 3.1.** For \(i = 1, 2\), let \(G_i = (G_i, (K_i^1), (\phi_e^i)_{e \in E})\) be two Mauldin-Williams graphs over the same underlying graph \(G\). Let \(A_i = C(K_i), i = 1, 2, \) be the associated C*-algebras and let \(X_i, i = 1, 2, \) be the associated C*-correspondences. If there is a homeomorphism \(f : K_1 \to K_2\), a partition of open subsets \(\{U_1, \ldots, U_m\}\) for \(K_1^1\), for some \(m \in \mathbb{N}\), and if for each \(U_j\) there is a permutation \(\sigma_j \in S_n\), where \(n = |E|\), such that \(f^{-1} \circ \phi_{\sigma_j(e)} \circ f|_{U_j} = \phi_{\sigma_j(e)}^1|_{U_j}\) and \(f(K_r^1) = K_r^2\) for all \(e \in E, j \in \{1, \ldots, m\}\), then \(X_1\) and \(X_2\) are isomorphic.

**Proof.** Since \(f\) is a homeomorphism, the map \(\beta : A_2 \to A_1\), defined by the equation \(\beta(b) = b \circ f\) for all \(b \in A_2\), is a C*-isomorphism. Define \(V : X_2 \to X_1\) by the formula
\[
V(\xi)(e, x) = \sum_{k=1}^{m} \xi_{\sigma_k(e)}(f(x)) \cdot 1_{U_k}(x),
\]
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for all $\langle e, x \rangle \in E \times G K$, where $\xi_{\sigma_k(e)}(f(x)) := \xi(\sigma_k(e), f(x))$. We show that $V$ is a $C^*$-correspondence isomorphism over $\beta$. Let $b_1, b_2 \in A_2$ and $\xi \in X_2$. We have

$$V(b_1 \cdot \xi \cdot b_2)(e, x) = \sum_{k=1}^{m} b_1 \circ \phi^2_{\sigma_k(e)}(f(x)) \xi_{\sigma_k(e)}(f(x)) b_2(f(x)) 1_{U_k}(x)$$

$$= \sum_{k=1}^{m} b_1 \circ \phi^1_{\sigma_k(e)}(f(x)) \xi_{\sigma_k(e)}(f(x)) 1_{U_k}(x) \cdot \beta(b_2)(x)$$

Also

$$\langle V(\xi), V(\eta) \rangle_{A_1}(x) = \sum_{f \in E \text{ if } \langle e, x \rangle \in K^2_{\sigma_k(e)}} \left( \sum_{k=1}^{m} \xi_{\sigma_k(e)}(f(x)) \eta_{\sigma_k(e)}(f(x)) 1_{U_k}(x) \right)$$

hence $\langle V(\xi), V(\eta) \rangle_{A_1} = \beta(\langle \xi, \eta \rangle_{A_2})$. Finally, one can see that $V$ is onto, hence $V$ is a $C^*$-correspondence isomorphism.

Recall that, for $k \geq 2$, $E^k := \{ \alpha = (\alpha_1, \cdots, \alpha_k) : \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}), i = 1, \cdots, k - 1 \}$, is the set of paths of length $k$ in the graph $G$. Let $E^* = \bigcup_{k \in \mathbb{N}} E^k$ be the space of finite paths in the graph $G$. Also the infinite path space, $E^\infty$, is defined to be

$$E^\infty := \{ (\alpha_i)_{i \in \mathbb{N}} : \alpha_i \in E \text{ and } r(\alpha_i) = s(\alpha_{i+1}) \text{ for all } i \in \mathbb{N} \}.$$

For $v \in V$, we also define $E^k(v) := \{ \alpha \in E^k : s(\alpha) = v \}$, and $E^*(v)$ and $E^\infty(v)$ are defined similarly. We consider $E^\infty(v)$ to be endowed with the metric: $\delta_v(\alpha, \beta) = d[\alpha \wedge \beta]$ if $\alpha \neq \beta$ and 0 otherwise, where $\alpha \wedge \beta$ is the longest common prefix of $\alpha$ and $\beta$, and $|w|$ is the length of the word $w \in E^*$ (see [5 Page 116]). Then $E^\infty(v)$ is a compact metric space, and, since $E^\infty$ equals the disjoint union of the spaces $E^\infty(v)$, $E^\infty$ becomes a compact metric space in a natural way. Define the maps $\theta_v : E^\infty(r(e)) \to E^\infty(s(e))$ by the formula $\theta_v(\alpha) = e \alpha$, for all $\alpha \in E^\infty$ and for all $e \in E$. Then $(G, (E^\infty(v))_{v \in V}, (\theta_v)_{v \in E})$ is a Mauldin-Williams graph. We set $A_E := C(E^\infty)$ and we set $E$ be the $C^*$-correspondence associated to this Mauldin-Williams graph. Let $M = (G, \{ K_v, \rho_v \}_{v \in E^0}, \{ \phi_v \}_{v \in E^1})$ be a Mauldin-Williams graph. For $(\alpha_1, \cdots, \alpha_n) \in E^n$ let $K_{(\alpha_1, \cdots, \alpha_n)} := \phi_{\alpha_1} \circ \cdots \phi_{\alpha_n}(K_{r(\alpha_n)})$. Then, for any infinite path $\alpha = (\alpha_n)_{n \in \mathbb{N}} \in E^\infty$, $\bigcap_{n \geq 1} K_{(\alpha_1, \cdots, \alpha_n)}$ contains only one point. Therefore, we can define a map $\pi : E^\infty \to K$ by $\{ \pi(x) \} = \bigcap_{n \geq 1} K_{(\alpha_1, \cdots, \alpha_n)}$. Since $\pi(E^\infty)$ is also an invariant set, $\pi$ is a continuous, onto map and $\pi(E^\infty(v)) = K_v$. Moreover, $\pi \circ \theta_v = \phi_v \circ \pi$.

We say that a Mauldin-Williams graph $\mathcal{M} = (G, \{ K_v, \rho_v \}_{v \in E^0}, \{ \phi_v \}_{v \in E^1})$ is totally disconnected if $\phi_v(K_s(e)) \cap \phi_f(K_t(f)) = \emptyset$ if $s(e) = s(f)$ and $e \neq f$.

**Corollary 3.2.** Let $\mathcal{M} = (G, \{ K_v, \rho_v \}_{v \in E^0}, \{ \phi_v \}_{v \in E^1})$ be a totally disconnected Mauldin-Williams graph. Let $A$ be the $C^*$-algebra and let $\mathcal{E}$ be the $C^*$-correspondence associated to this Mauldin-Williams graph. Then $\mathcal{E}$ is isomorphic with $\mathcal{E}$, as $C^*$-correspondences. In particular, one obtains that for any two totally disconnected Mauldin-Williams graphs having the same underlying graph $G$, the $C^*$-correspondences and tensor algebras associated are isomorphic.
Proof: If the Mauldin-Williams graph is totally disconnected, then the map \( \pi : E^\infty \to K \) defined above is a homeomorphism. Moreover, \( \pi \circ \theta_e \circ \pi^{-1} = \phi_e \) for all \( e \in E \), therefore the associated \( C^* \)-correspondences are isomorphic. \( \square \)

The converse of this corollary is true and will be proved later.

The next theorem is a converse of the Proposition 3.1. We note, however, that the family of open sets \( \{U_i\} \) here is not required to be a partition of the compact set \( K \), but only a finite open cover of it.

**Theorem 3.3.** For \( i = 1, 2 \), let \( G_i = (G, (K_i^e)_{e \in V}, (\phi_i^e)_{e \in E}) \) be two Mauldin-Williams graphs over the same underlying graph \( G \). Let \( A_i = C(K_i^1) \), \( i = 1, 2 \), be the associated \( C^* \)-algebras and let \( X_i, i = 1, 2 \), be the associated \( C^* \)-correspondences. If \( X_1 \) and \( X_2 \) are isomorphic, then there is a homeomorphism \( f : K^1 \to K^2 \), a finite open cover of \( K^1 \), \( \{U_1, \ldots, U_m\} \), and for each \( U_j \) there is a permutation \( \sigma_j \in S_n \) (\( n = |E| \)) such that

\[
(3.1) \quad f^{-1} \circ \phi_2^e \circ f|_{U_j} = \phi_1^e(g_j)|_{U_j} \quad \text{for all } e \in E, i \in \{1, \ldots, m\}.
\]

**Proof.** Since \( X_1 \) and \( X_2 \) are isomorphic, there is a \( C^* \)-isomorphism \( \beta : A_2 \to A_1 \) and a \( C^* \)-correspondence isomorphism \( W : X_2 \to X_1 \) such that \( W(b_1 \cdot \xi \cdot b_2) = \beta(b_1)W(\xi)\beta(b_2) \) and \( (W(\xi), W(\eta))_{A_1} = \beta((\xi, \eta)_{A_2}) \), for all \( b_1, b_2 \in A_2, \, \xi, \eta \in X_2 \). Let \( f : K^1 \to K^2 \) be the homeomorphism which implements \( \beta \), that is, \( \beta(b) = b \circ f \), for all \( b \in A_2 \).

Let \( \delta_e \in X_2 \), defined by

\[
\delta_e(g, y) = \begin{cases} 1, & \text{if } e = g, \\ 0, & \text{otherwise}, \end{cases}
\]

for \( e \in E \), be the natural basis in \( X_2 \) and let \( (\delta_e^e)_{e \in E} \subset X_1 \) be the natural basis in \( X_1 \), which is defined similarly.

For \( \xi \in X_2, \xi = \sum_{g \in E} \delta_g \cdot \xi_g \), where \( \xi_g(y) = \xi(g, y) \) for all \( y \in K^2_{r(e)} \) and is 0 otherwise. With respect to the bases, we can write

\[
W(\xi) = W\left( \sum_{g \in E} \delta_g \cdot \xi_g \right) = \sum_{g \in E} W(\delta_g) \cdot \xi_g \circ f
\]

\[
= \sum_{g \in E} \sum_{e \in E} \delta_e^e \cdot w_{eg} \xi_g \circ f,
\]

where

\[
W(\delta_g) = \sum_{e \in E} \delta_e^e \cdot w_{eg}, \quad w_{eg} \in A_1,
\]

and \( w_{eg} \) are given by the formula \( w_{eg} = \langle \delta_e^e, W(\delta_g) \rangle_{A_1} \), for all \( e, g \in E \). We call \( (w_{eg})_{e, g \in E} \) the matrix of \( W \) with respect to the basis \( (\delta_e^e)_{e \in E} \) and \( (\delta_g)_{g \in E} \) (it is an \( n \times n \) matrix, where \( n = |E| \)). Since \( W \) preserves the inner product, we see that

\[
(3.4) \quad \langle W(\delta_g), W(\delta_e) \rangle = \langle \delta_g, \delta_e \rangle = \delta_{ge},
\]

where \( \delta_{ge}(x) = 1 \) if \( e = g \) and \( x \in K_{r(e)} \) and is 0 otherwise. Also,

\[
(3.5) \quad \langle W(\delta_g), W(\delta_e) \rangle = \sum_{f \in E} w_{fg}^f w_{fe},
\]
Equations (3.4) and (3.5) imply that for every \( x \in K^n \) the matrix \( \begin{pmatrix} w_{xf}(x) \end{pmatrix}_{e, f \in E} \) is invertible. Hence there is \( \sigma_x \in S_n \) such that \( w_{\sigma_x(e),f}(x) \neq 0 \) for all \( e \in E \). Therefore, there is a neighborhood \( U_x \) of \( x \) such that

\[
(3.6) \quad w_{\sigma_x(e),f}(x) \neq 0 \quad \text{for all} \quad e \in E, \ y \in U_x \text{ and } x \in K^n.
\]

Let \( b \in A_2 \). Then, for \( h \in E \) we have that

\[
W(b \cdot \delta_h) = \sum_{e \in E} \delta^e_c w_{eh} b \circ \phi_h^2 \circ f
\]

and

\[
\beta(b) \cdot W(\delta_h) = \sum_{e \in E} \delta^e_c b \circ f \circ \phi_h^1 w_{eh}.
\]

Fix \( x \in K^n \) and let \( \sigma_x \in S_n \) and \( U_x \) be defined as in Equation (3.6). Then

\[
W(b \cdot \delta_h)(\sigma_x(h), y) = w_{\sigma_x(h),f}(y) b \circ \phi_h^2 \circ f(y)
\]

and

\[
(\beta(b) \cdot W(\delta_h))(\sigma_x(h), y) = b \circ f \circ \phi_h^1 w_{\sigma_x(h),f}(y)
\]

for all \( y \in U_x \) and for all \( h \in E \). Since \( W \) is a \( C^* \)-correspondence isomorphism and \( w_{\sigma_x(h),f}(y) \neq 0 \) for all \( y \in U_x \), for any \( x \in K^n \), there is a neighborhood \( U_x \) of \( x \) in \( K^n \) and there is a permutation \( \sigma_x \in S_n \) such that

\[
f^{-1} \circ \phi_h^2 \circ f \big|_{U_x} = \phi_h^{1 \sigma_x(h)} \big|_{U_x} \quad \text{for all} \quad h \in E.
\]

Hence we can find a finite cover \( \{U_1, \ldots, U_m\} \) of \( K^n \) and for each \( U_i \) we can find a permutation \( \sigma_i \in S_n \) such that the Equation (3.1) holds.

In the special case when the two Mauldin-Williams graphs are totally disconnected, more can be said about the choice of the permutations \( \sigma_i \).

**Corollary 3.4.** Let \( G_i = (G_i, (K_1^i)_{e \in E}, (\phi^1_e)_{e \in E}) \) be two Mauldin-Williams graphs. Let \( A_i = C(K^i) \) and let \( \mathcal{X}_i \), \( i = 1, 2 \), be the associated \( C^* \)-algebras and \( C^* \)-correspondences. If \( G_i \) is totally disconnected and if \( \mathcal{X}_1 \) is isomorphic with \( \mathcal{X}_2 \) there is a continuous map \( h : K^n \rightarrow S_n \) such that \( f^{-1} \circ \phi^2_h \circ f(x) = \phi_{h(x)}(e) \) for all \( x \in K^n \).

**Proof.** Recall that if \( G_i \) is totally disconnected, then \( \phi^1_e(K_{r(e)}) \cap \phi^1_f(K_{r(f)}) = \emptyset \) if \( e \neq f \). From the Theorem 3.3 we know that there are open sets \( \{U_1, \ldots, U_m\} \), for some \( m \in \mathbb{N} \), and permutations \( \sigma_1, \ldots, \sigma_m \in S_n \) such that

\[
(3.7) \quad f^{-1} \circ \phi^2_h \circ f \big|_{U_i} = \phi^1_{\sigma_i(h)} \big|_{U_i} \quad \text{for all} \quad e \in E, \ i \in \{1, \ldots, m\}.
\]

If \( U_i \cap U_j \neq \emptyset \) for some \( i \neq j \), then it follows that \( \phi^1_{\sigma_i(e)}(U_i \cap U_j) = \phi^1_{\sigma_j(e)}(U_i \cap U_j) \) for all \( e \in E \), hence \( \sigma_i(e) = \sigma_j(e) \) for all \( e \in E \), so \( \sigma_i = \sigma_j \). Therefore, we can choose the cover \( U_1, \ldots, U_m \) such that \( U_i \cap U_j = \emptyset \) if \( i \neq j \).

Let \( x \in K^n \). Then there is a unique \( i \in \{1, \ldots, n\} \) such that \( x \in U_i \). We define \( h(x) = \sigma_i \). Then \( h : K^n \rightarrow S_n \) is a well-defined map. Moreover, \( h \) is continuous (considering \( S_n \) endowed with the discrete topology), since for every \( \sigma \in S_n \), \( h^{-1}(\sigma) = \emptyset \) or \( h^{-1}(\sigma) = U_i \), for some \( i \in \{1, \ldots, n\} \). Finally, from the Equation (3.7) we obtain that

\[
f^{-1} \circ \phi^2_h \circ f(x) = \phi^1_{h(x)}(e) \quad \text{for all} \quad x \in K^n \text{ and } e \in E.
\]
Suppose that $G_i = (G, (K_i^1)_{v \in V}, (\phi_i^e)_{e \in E})$ are two Mauldin-Williams graphs that satisfy the hypothesis of the Corollary. We claim that $G_2$ must also be totally disconnected. Suppose that there are $e, f \in E$, $e \neq f$, such that $\phi_2^e(K_2^1(e)) \cap \phi_2^f(K_2^1(f)) \neq \emptyset$. Then there is an $x \in K^1$ such that $y := f(x) \in \phi_2^e(K_2^1(e)) \cap \phi_2^f(K_2^1(f))$. Then $\phi_2^1(K_2^1(x)(x)) = f(K_2^1(x)(y))$, which is a contradiction since $G_1$ is totally disconnected. So $G_2$ is totally disconnected.

**Example 3.5.** Let $K$ be the Cantor set, let $\phi_i : K \to K$, $i = 1, 2$, be the maps defined by the formulae $\phi_1(x) = \frac{1}{3}x$ and $\phi_2(x) = \frac{1}{3}x + \frac{2}{3}$. Then $T$ is the invariant set of $(\phi_1, \phi_2)$. Let $T = \{0, 1\}$ and let $\psi : T \to T$, $i = 1, 2$, be the maps defined by the formulae $\psi_1(x) = x$ and $\psi_2(x) = -\frac{1}{2}x + 1$. Then $T$ is the invariant set of $(\psi_1, \psi_2)$. Since $(\psi_1, \psi_2)$ is not totally disconnected, we see that the associated $C^*$-correspondences are not strongly Morita equivalent. Hence the tensor algebras fail to be strongly Morita equivalent in the sense of [2], yet their $C^*$-envelopes coincide with $\Omega_2$.

**Example 3.6.** Let $T$ be the regular triangle in $\mathbb{R}^2$ with vertices $A = (0, 0)$, $B = (1, 0)$ and $C = (1/2, \sqrt{3}/2)$. Let $\phi_1(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{3} + \frac{1}{2}\right)$, $\phi_2(x, y) = \left(\frac{x}{2} + \frac{1}{2}, \frac{y}{3}\right)$ and $\phi_3(x, y) = \left(\frac{x}{3} + \frac{1}{2}, \frac{y}{3}\right)$. Then the invariant set $K$ of this iterated function system is the Sierpinski gasket. Let $\psi_1 = \sigma_1 \circ \phi_1$, $\psi_2 = \phi_2$ and $\psi_3 = \phi_3$, where $\sigma_i$ is the symmetry about the median from the vertex $\phi_i(C)$ of the triangle $\phi_i(T)$, for $i = 1, 3$. Then the invariant set of this iterated function system is also the Sierpinski gasket, but the $C^*$-correspondences associated to $(\phi_1, \phi_2, \phi_3)$ and $\psi_1, \psi_2, \psi_3$ are not isomorphic.

**References**


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