IRREDUCIBLE REPRESENTATIONS OF GROUPOID C^*-ALGEBRAS

MARIUS IONESCU AND DANA P. WILLIAMS

ABSTRACT. If G is a second countable locally compact Hausdorff groupoid with Haar system, we show that every representation induced from an irreducible representation of a stability group is irreducible.

1. INTRODUCTION

To understand the fine structure of a C^*-algebra, a good first step is to describe the primitive ideals in a systematic way. Therefore producing a prescription for a robust family of irreducible representations is very important. In the case of transformation group C^*-algebras, it is well known that representations induced from irreducible representations of the stability groups are themselves irreducible. (Furthermore, in many cases these representations exhaust the irreducible representations, or at least their kernels exhaust the collection of primitive ideals, and a fairly complete description of the primitive ideal space is possible. For a more extensive discussion, see [20, §§8.2–3].) In the separable case, the irreducibility of representations induced from irreducible representations of the stability groups is due to Mackey [9, §6] (see also Glimm's [5, pp. 900–901]). The result for general transformation group C^*-algebras was proved in [19, Proposition 4.2] (see also [20, Proposition 8.27]). The corresponding result for groupoid C^*-algebras has been proved in an *ad hoc* manner in a number of special cases (see Example 3 for specific references). In this note, we want to prove the result for general *separable* groupoids. In so doing, we take the opportunity to formalize the theory of inducing representations from a general closed subgroupoid. Of course, induction is treated in Renault's thesis [16, Chap. II §2]. However, at the time [16] was written, Renault did not yet have the full power of his disintegration theorem ([17, Proposition 4.2] or see [14, Proposition 7.8]) available. Nor was Rieffel's theory of Morita equivalence fully developed. So it seems appropriate to give a modern treatment here using a contemporary version of Rieffel's theory, and the disintegration theorem in the form of the equivalence theorem from [10, Theorem 2.8].

In Section 2, we derive the general process for inducing groupoid representations of a closed subgroupoid H of G to G — actually, we induce representations from $C^*(H)$ to $C^*(G)$. In Section 3, we specialize to the case where H is an isotropy group, $H = G(u) = G_u^u := \{\, x \in G : s(x) = r(x) \,\}$, and prove the main result.

Throughout, G will be a second countable locally compact Hausdorff groupoid. Second countability, in the form of the separability of $C^*(G)$, is necessary in Section 2 in order to invoke the disintegration theorem. Although separability might be unnecessary in the proof of the main theorem, we felt that, as much of the deep theory of groupoid C^*-algebras uses the disintegration result in one form or another, there was little to be gained which would justify the additional work of adjusting the proof to handle the general case. In addition, we always assume that G and H have (continuous) Haar systems. We adopt the usual conventions that representations of C^*-algebras are nondegenerate and that homomorphisms between C^*-algebras are necessarily $*$-preserving.

2. Inducing Representations

We assume that G is a second countable locally compact groupoid with Haar system $\{\, \lambda^u \,\}_{u \in G^{(0)}}$. Let H be a closed subgroupoid of G with Haar system $\{\, \alpha^u \,\}_{u \in H^{(0)}}$. Since H is closed in G, we also have $H^{(0)}$ closed in $G^{(0)}$. Then $G_{H^{(0)}} := s^{-1}(H^{(0)})$ is a locally compact free and proper right H-space and we can form the imprimitivity groupoid H^G as follows. The space

$$G_{H^{(0)}} *_s G_{H^{(0)}} := \{\, (x, y) \in G_{H^{(0)}} \times G_{H^{(0)}} : s(x) = s(y) \,\}$$

is a free and proper right H-space for the diagonal action $(x, y) \cdot h := (xh, yh)$. Consequently, the orbit space

$$H^G := (G_{H^{(0)}} *_s G_{H^{(0)}})/H$$

is a locally compact Hausdorff space. Following [10, §2], H^G is a groupoid in a natural way. If $[x, y]$ denotes the orbit of (x, y) in H^G, then the composable pairs are given by

$$(H^G)^{(2)} := \{\, \big([x, y], [z, w]\big) : y \cdot H = z \cdot H \,\},$$

and the groupoid operations are given by

$$[x, y][yh, z] := [x, zh^{-1}] \quad \text{and} \quad [x, y]^{-1} := [y, x].$$

We can identify $(H^G)^{(0)}$ with $G_{H^{(0)}}/H$ and then

$$r\big([x, y]\big) = x \cdot H \quad \text{and} \quad s\big([x, y]\big) = y \cdot H.$$

It is not hard to check that H^G acts freely and properly on the left of $G_{H^{(0)}}$:

$$[x, y] \cdot (yh) = xh,$$

and that $G_{H^{(0)}}$ is then a (H^G, H)-equivalence as in [10, Definition 2.1].

To get a Haar system on H^G, we proceed as in [7, §5]. Since $G_{H^{(0)}}$ is closed in G and since λ is a Haar system on G, it is not hard to check that

$$\beta'(\varphi)(x) := \int_G \varphi(y) \, d\lambda_{s(x)}(y) \quad \varphi \in C_c(G_{H^{(0)}})$$

is a full equivariant s-system for the map $s : G_{H^{(0)}} \to H^{(0)}$ (as defined in [7, §5]). Therefore [7, Proposition 5.2] implies that we get a Haar system $\{\, \beta^{x \cdot H} \,\}_{x \cdot H \in (H^G)^{(0)}}$ for H^G via

$$\beta(F)(x \cdot H) = \int_{H^G} F\big([x, y]\big) \, d\beta^{x \cdot H}\big([x, y]\big) = \int_G F\big([x, y]\big) \, d\lambda_{s(x)}(y).$$

Since both H and H^G have Haar systems, [10, Theorem 2.8] implies that $C_c(G_{H^{(0)}})$ is a pre-$C_c(H^G, \beta) - C_c(H, \alpha)$-imprimitivity bimodule with actions and inner products given by

$$(1) \qquad\qquad F \cdot \varphi(z) = \int_G F([x, y])\varphi(y)\, d\lambda_{s(z)}(y)$$

$$(2) \qquad\qquad \varphi \cdot g(z) = \int_H \varphi(zh)g(h^{-1})\, d\alpha^{s(z)}(h)$$

$$(3) \qquad\qquad \langle \varphi\,,\,\psi \rangle_\star(h) = \int_G \overline{\varphi(y)}\psi(yh)\, d\lambda_{r(h)}(y)$$

$$(4) \qquad\qquad {}_\star\langle \varphi\,,\,\psi \rangle([x, y]) = \int_H \varphi(yh)\overline{\psi(xh)}\, d\alpha^{s(x)}(h).$$

We will write $\mathsf{X} = \mathsf{X}_H^G$ for the completion of $C_c(G_{H^{(0)}})$ as a $C^*(H^G) - C^*(H)$-imprimitivity bimodule.

If L is a representation of $C^*(H, \alpha)$, then we write X–Ind L for the representation of $C^*(H^G, \beta)$ induced via X (see the discussion following [15, Proposition 2.66]). Recall that X–Ind L acts on the completion $\mathcal{H}_{\mathrm{Ind}\,L}$ of $C_c(G_{H^{(0)}}) \odot \mathcal{H}_L$ with respect to the pre-inner product given on elementary tensors by

$$(\varphi \otimes h \mid \psi \otimes k) = \big(L(\langle \psi\,,\,\varphi \rangle_\star)h \mid k\big).$$

If $\varphi \otimes_H h$ denotes the class of $\varphi \otimes h$ in $\mathcal{H}_{\mathrm{Ind}\,L}$, then

$$(\mathsf{X}\text{–Ind}\,L)(F)(\varphi \otimes_H h) = F \cdot \varphi \otimes_H h.$$

To get an induced representation of $C^*(G)$ out of this machinery (i.e., using [15, Proposition 2.66]), we need a nondegenerate homomorphism of $C^*(\mathsf{G})$ into $\mathcal{L}(\mathsf{X})$. If $f \in C_c(G)$ and $\varphi \in C_c(G_{H^{(0)}})$ we can define

$$(5) \qquad\qquad f \cdot \varphi(z) = \int_G f(y)\varphi(y^{-1}z)\, d\lambda^{r(z)}(y).$$

Remark 1. Since $G_{H^{(0)}}$ is closed in G, each $\varphi \in C_c(G_{H^{(0)}})$ is the restriction of an element $f_\varphi \in C_c(G)$. Thus we can write

$$\langle \varphi\,,\,\psi \rangle_\star = \varphi * \psi \quad\text{and}\quad f \cdot \varphi = f * \varphi,$$

where, for example, $\varphi * \psi$ should be interpreted as $f_\varphi * f_\psi$ restricted to $G_{H^{(0)}}$ — the point being that the restriction is independent of our choice of f_φ and f_ψ. Similarly, $f * \varphi$ is meant to be the restriction of $f * f_\varphi$ to $G_{H^{(0)}}$.

If ρ is a state on $C^*(H)$, then

$$(\cdot \mid \cdot)_\rho := \rho(\langle \cdot\,,\,\cdot \rangle_\star)$$

is a pre-inner product on $C_c(G_{H^{(0)}})$ with Hilbert space completion denoted by $\mathcal{H}_\rho$. If we let $V(f)\varphi := f \cdot \varphi$, then it follows from direct computation, or by invoking Remark 1 above, that

$$\langle V(f)\varphi\,,\,\psi \rangle_\star = \langle f \cdot \varphi\,,\,\psi \rangle_\star = \langle \varphi\,,\,f^* \cdot \psi \rangle_\star = \langle \varphi\,,\,V(f^*)\psi \rangle_\star.$$

Thus V induces a map of $C_c(G)$ into the linear operators on the dense image of $C_c(G_{H^{(0)}})$ in $\mathcal{H}_\rho$ which clearly satisfies the axioms of Renault's disintegration theorem (see, e.g., [14, Theorem 7.8] or [17, Proposition 4.2]). In particular, we obtain a bona fide representation of $C^*(G)$ on $\mathcal{H}_{\mathrm{Ind}\,L}$, and it follows that

$$\rho(\langle f \cdot \varphi\,,\,f \cdot \varphi \rangle_\star) \leq \|f\|^2_{C^*(G)}\rho(\langle \varphi\,,\,\varphi \rangle_\star).$$

Since this holds for all ρ,

$$\langle f \cdot \varphi \, , \, f \cdot \varphi \rangle_\star \leq \|f\|^2 \langle \varphi \, , \, \varphi \rangle_\star,$$

and $V(f)$ is a bounded adjointable operator on X. Therefore we obtain an induced representation $\mathrm{Ind}_H^G L$ of $C^*(G)$ on $\mathcal{H}_{\mathrm{Ind}\, L}$ such that

$$(\mathrm{Ind}_H^G L)(f)(\varphi \otimes_H h) = f * \varphi \otimes_H h.$$

Remark 2. Since $M\big(C^*(H^G)\big) \cong \mathcal{L}(\mathsf{X})$ ([15, Corollary 2.54 and Proposition 3.8]), it is not hard to see that $\mathrm{Ind}_H^G L$ is the composition of V with the natural extension of X–$\mathrm{Ind}\, L$ to $\mathcal{L}(\mathsf{X})$.

Example 3. In the next section, we will be exclusively interested in the special case of the above where H is the stability group at a $u \in G^{(0)}$. That is,

(6) $$H = G(u) := G_u^u = \{\, x \in G : s(x) = u = r(x) \,\}.$$

(Thus, $H^{(0)} = \{\, u \,\}$.) In this case, we obtain the induced representations used to establish special cases of Theorem 5 in [1, Lemma 4.2; 4, §5; 11, Lemma 2.5; 12, Lemma 2.4; 13, Lemma 3.2].

One advantage of having a formal theory of induction for representations of groupoid C^*-algebras is that we can apply the Rieffel machinery. An example is the following version of induction in stages. The proof, modulo technicalities, is a straightforward modification of Rieffel's original "C^*-version" from [18, Theorem 5.9]. For future reference, we've worked out the details of the proof in the last section.

Theorem 4 (Induction in Stages)**.** *Suppose that H and K are closed subgroupoids of a second countable locally compact Hausdorff groupoid G with $H \subset K$. Assume that H, K and G have Haar systems. If L is a representation of $C^*(H)$, then*

$$\mathrm{Ind}_H^G L \quad and \quad \mathrm{Ind}_K^G\big(\mathrm{Ind}_H^K L\big)$$

are equivalent representations of $C^*(G)$.

3. The Main Theorem

Theorem 5. *Let G be a second countable groupoid with Haar system $\big\{\, \lambda^u \,\big\}_{u \in G^{(0)}}$. Suppose that L is an irreducible representation of the stability group $G(u)$ at $u \in G^{(0)}$. Then $\mathrm{Ind}_{G(u)}^G L$ is an irreducible representation of $C^*(G)$.*

The idea of the proof is straightforward. Let L be an irreducible representation of $C^*\big(G(u)\big)$. Since X is a $C^*\big(G(u)^G\big)$–$C^*\big(G(u)\big)$-imprimitivity bimodule, [15, Corollary 3.32] implies that X–$\mathrm{Ind}\, L$ is an irreducible representation of $C^*\big(G(u)^H\big)$. We will show that any T in the commutant of $\mathrm{Ind}_{G(u)}^G L$ is a scalar multiple of the identity. It will suffice to see that any such T commutes with $(\mathsf{X}$–$\mathrm{Ind}\, L)(F)$ for all $F \in C_c\big(G(u)^G\big)$. Our proof will consist in producing, given F, a net $\{\, f_i \,\}$ in $C_c(G)$ such that

$$(\mathrm{Ind}_{G(u)}^G L)(f_i) \to (\mathsf{X}\text{–}\mathrm{Ind}\, L)(F)$$

in the weak operator topology. Since we will also arrange that this net is uniformly bounded in the $\| \cdot \|_I$-norm on $C_c(G)$ — so that the net $\big\{\, (\mathrm{Ind}_{G(u)}^G L)(f_i) \,\big\}$ is uniformly bounded in $B\big(\mathcal{H}_{\mathrm{Ind}\, L}\big)$ — we just have to arrange that

$$\big((\mathrm{Ind}_{G(u)}^G L)(f_i)(\varphi \otimes_{G(u)} h) \mid \psi \otimes_{G(u)} k\big) \to \big((\mathsf{X}\text{–}\mathrm{Ind}\, L)(F)(\varphi \otimes_{G(u)} h) \mid \psi \otimes_{G(u)} k\big)$$

for all $\varphi, \psi \in C_c(G_u)$ and $h, k \in \mathcal{H}_L$.

The next lemma is the essential ingredient to our proof.

Lemma 6. *Suppose that* $F \in C_c\big(G(u)^G\big)$. *Then there is compact set* C_F *in* G *such that for each compact set* $K \subset G_u$ *there is a* $f_K \in C_c(G)$ *such that*

(a) $f_K(zy^{-1}) = F\big([z, y]\big)$ *for all* $(z, y) \in K \times K$,
(b) $\operatorname{supp} f_K \subset C_F$ *and*
(c) $\|f_K\|_I \leq \|F\|_I + 1$.

The proof of Lemma 6 is a bit technical, so we'll postpone the proof for a bit, and show that the lemma allows us to prove Theorem 5.

Proof of Theorem 5. For each $K \subset G_u$, let f_K be as in Lemma 6. Then $\{ f_K \}$ and $\{ (\operatorname{Ind}_{G(u)}^G L)(f_K) \}$ are nets indexed by increasing K. Notice that

(7)
$$\big((\operatorname{Ind}_{G(u)}^G L)(f_K)(\varphi \otimes_{G(u)} h) \mid \psi \otimes_{G(u)} k\big) -$$
$$\big((\mathsf{X}\text{–Ind } L)(F)(\varphi \otimes_{G(u)} h) \mid \psi \otimes_{G(u)} k\big)$$
$$= \big(L\big(\langle \psi \,,\, f_K * \varphi - F \cdot \varphi \rangle_\star\big)h \mid k\big)$$

Furthermore, using the invariance of the Haar system on G, we can compute as follows:

(8)
$$\langle \psi \,,\, f_K * \varphi \rangle_\star (s) = \int_G \overline{\psi(x)} f_K * \varphi(xs) \, d\lambda_u(x)$$
$$= \int_G \int_G \overline{\psi(x)} f_K(xz^{-1}) \varphi(zs) \, d\lambda_u(z) \, d\lambda_u(x),$$

while on the other hand,

(9)
$$\langle \psi \,,\, F \cdot \varphi \rangle_\star (s) = \int_G \overline{\psi(x)} F \cdot \varphi(xs) \, d\lambda_u(x)$$
$$= \int_G \int_G \overline{\psi(x)} F\big([xs, z]\big) \varphi(z) \, d\lambda_u(z) \, d\lambda_u(x)$$
$$= \int_G \int_G \overline{\psi(x)} F\big([x, zs^{-1}]\big) \varphi(z) \, d\lambda_u(z) \, d\lambda_u(x)$$
$$= \int_G \int_G \overline{\psi(x)} F\big([x, z]\big) \varphi(z) \, d\lambda_u(zs) \, d\lambda_u(x)$$

Notice that $\operatorname{supp}\langle \psi \,,\, \varphi \rangle_\star \subset (\operatorname{supp} \psi)(\operatorname{supp} \varphi)$. Since $\operatorname{supp} f_K \subset C_F$ for all K, we have

$$\operatorname{supp} f_K * \varphi \subset (\operatorname{supp} f_K)(\operatorname{supp} \varphi) \subset C_F(\operatorname{supp} \varphi).$$

Therefore if (8) does not vanish, then we must have $s \in (\operatorname{supp} \psi) C_F(\operatorname{supp} \varphi)$. Therefore there is a compact set K_0 — which does *not* depend on K — such that both (8) and (9) vanish if $s \notin K_0$. Thus if $s \in K_0$ and if $K \supset (\operatorname{supp} \psi) \cup (\operatorname{supp} \varphi)K_0^{-1}$, then the integrand in (8) is either zero or we must have $(x, z) \in K \times K$. Therefore we can replace $f_K(xz^{-1})$ by $F\big([x, z]\big)$ and $f_K * \varphi - F \cdot \varphi$ is the zero function whenever K contains $(\operatorname{supp} \psi) \cup (\operatorname{supp} \varphi)K_0^{-1}$. Therefore the left-hand side of (7) is eventually zero, and the theorem follows. $\qquad\square$

We still need to prove Lemma 6, and to do that, we need some preliminaries. In the sequel, if S is a Borel subset of G, then

$$\int_S f(x)\, d\lambda^u(x) := \int_G \mathbb{1}_S(x) f(x)\, d\lambda^u(x)$$

where $\mathbb{1}_S$ is the characteristic function of S.

Lemma 7. *Suppose that $f \in C_c^+(G)$ and that $K \subset G$ is a compact set such that*

$$\int_K f(x)\, d\lambda^u(x) \le M \quad \text{for all } u \in G^{(0)}.$$

There there is a neighborhood V of K such that

$$\int_V f(x)\, d\lambda^u(x) \le M + 1 \quad \text{for all } u \in G^{(0)}.$$

Proof. Let K_1 be a compact neighborhood of K. Since G is second countable, we can find a countable fundamental system $\{V_n\}$ of neighborhoods of K in K_1; thus, given any neighborhood V of K, there is a n such that $V_n \subset V$ and $K = \bigcap V_n$. Certainly, we can assume that $V_{n+1} \subset V_n$.

If no V as prescribed in the lemma exists, then for each n we can find $u_n \in G^{(0)}$ such that

$$\int_{V_n} f(x)\, d\lambda^{u_n}(x) \ge M + 1.$$

Since we must have each $u_n \in r(K_1)$, we can pass to a subsequence, relabel, and assume that $u_n \to u_0$. Since $\mathbb{1}_{V_n} \to \mathbb{1}_K$ pointwise, the dominated convergence theorem implies that

$$\int_{V_n} f(x)\, d\lambda^{u_0}(x) \to \int_K f(x)\, d\lambda^{u_0}(x).$$

In particular, there is a n_1 such that

$$\int_{V_{n_1}} f(x)\, d\lambda^{u_0}(x) \le M + \frac{1}{2}$$

Let W be an open set such that $K \subset W \subset \overline{W} \subset V_{n_1}$, and let $f_0 \in C_c^+(G)$ be such that $f_0|_{\overline{W}} = f$, $f_0 \le f$ and $\operatorname{supp} f_0 \subset V_{n_1}$. Then

$$\int_G f_0(x)\, d\lambda^{u_0}(x) \le M + \frac{1}{2}.$$

However, since $\{\lambda^u\}$ is a Haar system,

$$\int_G f_0(x)\, d\lambda^{u_n}(x) \to \int_G f_0(x)\, d\lambda^{u_0}(x) \le M + \frac{1}{2}.$$

But for large n, we have $V_n \subset W$ and therefore

$$\int_G f_0(x)\, d\lambda^{u_n}(x) \ge \int_{V_n} f_0(x)\, d\lambda^{u_n}(x) = \int_{V_n} f(x)\, d\lambda^{u_n}(x) \ge M + 1.$$

This leads to a contradiction and completes the proof of the lemma. $\square$

Proof of Lemma 6. The map $(z, y) \mapsto zy^{-1}$ is certainly continuous on $G_u \times G_u$ and factors through the orbit map $\pi : G_u \times G_u \to G(u)^G$. In fact, if $zy^{-1} = xw^{-1}$, then we must have $z = x(w^{-1}y)$ and $y = w(x^{-1}z)$. But $w^{-1}y = x^{-1}z$ and lies in $G(u)$. Therefore, we have a well-defined injection $\Pi : G(u)^G \to G$ sending $[z, y]$ to zy^{-1}. We let C_F be a compact neighborhood of $\Pi(\operatorname{supp} F)$.

Fix a compact set $K \subset G_u$. The restriction of Π to the compact set $\pi(K \times K)$ is a homeomorphism so we can find a function $\tilde{f}_K \in C_c(G)$ such that $\operatorname{supp} \tilde{f}_K \subset C_F$ and such that $\tilde{f}_K(zy^{-1}) = F([z, y])$ for all $(z, y) \in K \times K$.

Let $K_G := \pi(K \times K)$. If

$$\int_{K_G} |\tilde{f}_K(y)| \, d\lambda^w(y) \neq 0,$$

then $K_G \cap G^w \neq \emptyset$. Thus there is a $z \in K$ such that $r(z) = w$ (and $s(z) = u$). Then by left invariance

$$\int_{K_G} |\tilde{f}_K(y)| \, d\lambda^w(y) = \int_G \mathbb{1}_{K_G}(zy) |\tilde{f}_K(zy)| \, d\lambda^u(y)$$

$$= \int_G \mathbb{1}_{K_G}(zy^{-1}) |\tilde{f}_K(zy^{-1})| \, d\lambda_u(y)$$

$$= \int_G \mathbb{1}_{K_G}(zy^{-1}) |F([z, y])| \, d\lambda_u(y)$$

$$\leq \|F\|_I.$$

Similarly, if

$$\int_{K_G} |\tilde{f}_K(y^{-1})| \, d\lambda^w(y) \neq 0,$$

then as before there is a $z \in K$ such that $r(z) = w$ and

$$\int_{K_G} |\tilde{f}_K(y^{-1})| \, d\lambda^w(y) = \int_G \mathbb{1}_{K_G}(zy) |\tilde{f}_K(y^{-1}z^{-1})| \, d\lambda^u(y)$$

$$= \int_G \mathbb{1}_{K_G}(zy^{-1}) |\tilde{f}_K(yz^{-1})| \, d\lambda_u(y)$$

which, since $K_G^{-1} = K_G$, is

$$= \int_G \mathbb{1}_{K_G}(yz^{-1}) |\tilde{f}_K(yz^{-1})| \, d\lambda_u(y)$$

$$\leq \int_G |F([y, z])| \, d\lambda_u(y)$$

$$= \int_{G(u)^G} |F([z, y]^{-1})| \, d\beta^{z \cdot G(u)}([z, y])$$

$$\leq \|F\|_I.$$

Using Lemma 7, we can find a neighborhood V of K_G contained in C_F such that both

$$\int_V |\tilde{f}_K(x)| \, d\lambda^w(x) \quad \text{and} \quad \int_V |\tilde{f}_K(x^{-1})| \, d\lambda^w(x)$$

are bounded by $\|F\|_I + 1$ for all $w \in G^{(0)}$. Since K_G is symmetric, we can assume that $V = V^{-1}$ as well. We can now let f_K be any element of $C_c(G)$ such that $f_K = \tilde{f}_K$ on K_G, $\operatorname{supp} f_K \subset V$ and $f_K \leq \tilde{f}_K$ everywhere. Then $\|f\|_I \leq \|F\| + 1$, $\operatorname{supp} f_K \subset C_F$ and $f_K(yz^{-1}) = F([z, y])$ for all $(z, y) \in K \times K$. This completes the proof of the lemma. $\qquad \square$

Example 8 (Holonomy Groupoid). Let $(V, \mathcal{F})$ be a C^∞ compact foliated manifold, and let G be its holonomy groupoid [6, 21] equipped with its usual locally compact topology as in [2]. Naturally, we also assume that G is Hausdorff so that our

results apply. The stability groups $G(x)$ for $x \in V$ are the holonomy groups for the foliation. Using Theorem 5, each irreducible representation σ_x of $G(x)$ provides an irreducible representation $\operatorname{Ind}_{G(x)}^G \sigma_x$ of $C^*(G)$. This representation is equivalent to the representations $\operatorname{Ind}_x \sigma_x$ treated in [4, §5]. Thus we recover a part of [4, Corollaire 5.7].

Example 9. If $\sigma : X \to X$ is a covering map for a compact Hausdorff space X, then the associated *Deaconu-Renualt groupoid* [3] is

$$G := \{\, (z, n - l, w) \in X \times \mathbf{Z} \times X : \sigma^l(z) = \sigma^n(w) \,\}.$$

More concretely, we can let X be the circle $\mathbf{T}$ and $\sigma(z) := z^2$. Then the stability group $G(z)$ at $(z, 0, z)$ is trivial unless $z = 1$ or z is a primitive 2^n-th root of unity. Then $G(1) = \{\, 1, k, 1) : k \in \mathbf{Z} \,\}$ and if z is a primitive 2^n-th root of unity, then $G(z) = \{\, (z, nk, z) : k \in \mathbf{Z} \,\}$. Now applying Theorem 5, we see that if z is not a primitive root, $\operatorname{Ind}_{G(z)}^G \delta(z, 0, z)$ is an irreducible regular representation. If z is a primitive 2^n-th root of unity, then for each $\omega \in \mathbf{T} \cong \widehat{\mathbf{Z}}$, we obtain an irreducible representation $\operatorname{Ind}_{G(z)}^G \omega$.

4. Proof of Theorem 4

We let λ, β and α be Haar systems on G, K and H, respectively. It will be helpful to notice that the space $\mathcal{H}_{\operatorname{Ind} L}$ of $\operatorname{Ind}_H^G L$ is an internal tensor product $\mathsf{X}_H^G \otimes_H \mathcal{H}_L$ for the appropriate actions of $C^*(H)$.[1] Thus the space of $\operatorname{Ind}_K^G \big(\operatorname{Ind}_H^G L\big)$ is $\mathsf{X}_K^G \otimes_K (\mathsf{X}_H^K \otimes_H \mathcal{H}_L)$. Of course, the natural map on the algebraic tensor products induces an isomorphism U of $\mathsf{X}_K^G \otimes_K (\mathsf{X}_H^K \otimes_H \mathcal{H}_L)$ with $(\mathsf{X}_K^G \otimes_K \mathsf{X}_H^K) \otimes_H \mathcal{H}_L$ (see [20, Lemma I.6]). We need to combine this with the following observation.

Lemma 10. *The map sending* $\varphi \odot \psi \in C_c(G_{K^{(0)}}) \odot C_c(K_{H^{(0)}})$ *to* $\theta(\varphi \otimes \psi)$ *in* $C_c(G_{H^{(0)}})$, *given by*

$$\theta(\varphi \otimes \psi)(x) := \int_K \varphi(xk)\psi(k^{-1}) \, d\beta^{s(x)}(k),$$

induces an isomorphism, also called θ, *of* $\mathsf{X}_K^G \otimes \mathsf{X}_H^K$ *onto* X_H^G.

Proof. The first step is to see that θ is isometric. Notice that we have three sets of actions and inner products. We have not tried to invent notation to distinguish one from another. Instead, we will hope that it is "clear from context" which formula is being employed. In this spirit,

$$\langle \varphi_1 \otimes \psi_1 \,,\, \varphi_2 \otimes \psi_2 \rangle_\star(h) = \langle \psi_1 \,,\, \langle \varphi_1 \,,\, \varphi_2 \rangle_\star \cdot \psi_2 \rangle_\star(h)$$

which, using (3) on $C_c(K_{H^{(0)}})$, is

$$= \int_K \overline{\psi_1(k)} \langle \varphi_1 \,,\, \varphi_2 \rangle_\star \cdot \psi_2(kh) \, d\beta_{r(h)}(k)$$

which, using (5) for the $C_c(K)$-action on $C_c(K_{H^{(0)}})$, is

$$= \int_K \int_K \overline{\psi_1(k)} \langle \varphi_1 \,,\, \varphi_2 \rangle_\star (kk_1)\psi_2(k_1^{-1}k) \, d\beta^{r(h)}(k_1) \, d\beta_{r(h)}(k)$$

---

[1]Internal tensor products of Hilbert modules are discussed in [8; 20, App. I].

which, using (3), is

$$= \int_K \int_K \int_G \overline{\psi_1(k)\varphi_1(x)}\varphi_2(xkk_1)\psi_2(k_1^{-1}k)\,d\lambda_{r(k)}(x)\,d\beta^{r(h)}(k_1)\,d\beta_{r(h)}(k)$$
$$= \int_K \int_K \int_G \overline{\psi_1(k)\varphi_1(xk^{-1})}\varphi_2(xk_1)\psi_2(k_1^{-1}k)\,d\lambda_{r(h)}(x)\,d\beta^{r(h)}(k_1)\,d\beta_{r(h)}(k)$$

which, after using Fubini and sending k_1 to hk_1, is

$$= \int_G \overline{\theta(\varphi_1 \otimes \psi_1)(x)}\theta(\varphi_2 \otimes \psi_2)(xh)\,d\lambda_{r(h)}(x)$$
$$= \langle \theta(\varphi_1 \otimes \psi_1)\,,\,\theta(\varphi_2 \otimes \psi_2)\rangle_\star.$$

Thus, θ is isometric. We just need to see that it has dense range.

However, notice that $\theta(\varphi \otimes \psi) = \varphi \cdot g_\psi$ for the right action on $C_c(K)$ on $C_c(G_{K^{(0)}})$ with g_ψ any extension of ψ to $C_c(K)$ (see (2) and Remark 1). It follows from [10, Proposition 2.10] that there is an approximate identity for $C_c(K)$ such that $\varphi \cdot g_i \to \varphi$ in the inductive limit topology for all $\varphi \in C_c({}_{K^{(0)}})$. This implies that the range of θ is dense, and completes the proof of the lemma. $\square$

Proof of Theorem 4. Define a unitary $V : \mathsf{X}_K^G \otimes_K (\mathsf{X}_H^K \otimes_H \mathcal{H}_L) \to \mathsf{X}_H^G \otimes_H \mathcal{H}_L$ by $V = \theta \circ U$ (where U is defined prior to Lemma 10). Then on elementary tensors, $V\big(\varphi \otimes (\psi \otimes h)\big) = \theta(\varphi \otimes \psi) \otimes h$. Then on the one hand,

$$V\big(\mathrm{Ind}_K^G\big(\mathrm{Ind}_H^K(L)(f)\big)\big)\big(\varphi \otimes (\psi \otimes h)\big) = \theta(f * \varphi \otimes \psi) \otimes h.$$

On the other hand, $\theta(f * \varphi \otimes \psi) = f * \theta(\varphi \otimes \psi)$. Therefore

$$V\big(\mathrm{Ind}_K^G\big(\mathrm{Ind}_H^K(L)\big) = (\mathrm{Ind}_H^G L)V.$$

This completes the proof. $\square$

References

[1] Lisa Orloff Clark, *CCR and GCR groupoid C^*-algebras*, Indiana Univ. Math. J. **56** (2007), 2087–2110.

[2] Alain Connes, *A survey of foliations and operator algebras*, Operator algebras and applications, Part I (Kingston, Ont., 1980), Proc. Sympos. Pure Math., vol. 38, Amer. Math. Soc., Providence, R.I., 1982, pp. 521–628.

[3] Valentin Deaconu, *Groupoids associated with endomorphisms*, Trans. Amer. Math. Soc. **347** (1995), 1779–1786.

[4] T. Fack and G. Skandalis, *Sur les représentations et idéaux de la C^*-algèbre d'un feuilletage*, J. Operator Theory **8** (1982), 95–129.

[5] James Glimm, *Families of induced representations*, Pacific J. Math. **12** (1962), 885–911.

[6] André Haefliger, *Groupoïdes d'holonomie et classifiants*, Astérisque (1984), 70–97. Transversal structure of foliations (Toulouse, 1982).

[7] Alexander Kumjian, Paul S. Muhly, Jean N. Renault, and Dana P. Williams, *The Brauer group of a locally compact groupoid*, Amer. J. Math. **120** (1998), 901–954.

[8] E. Christopher Lance, *Hilbert C^*-modules: A toolkit for operator algebraists*, London Math. Soc. Lecture Note Series, vol. 210, Cambridge Univ. Press, Cambridge, 1994.

[9] George W. Mackey, *Imprimitivity for representations of locally compact groups. I*, Proc. Nat. Acad. Sci. U. S. A. **35** (1949), 537–545.

[10] Paul S. Muhly, Jean N. Renault, and Dana P. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Operator Theory **17** (1987), 3–22.

[11] _____, *Continuous-trace groupoid C^*-algebras, III*, Trans. Amer. Math. Soc. (1996), 3621–3641.

[12] Paul S. Muhly and Dana P. Williams, *Continuous trace groupoid C^*-algebras*, Math. Scand. **66** (1990), 231–241.

[13] ———, *Continuous trace groupoid C^*-algebras. II*, Math. Scand. **70** (1992), 127–145.

[14] ———, *Renault's equivalence theorem for groupoid crossed products*, NYJM Monographs, vol. 3, State University of New York University at Albany, Albany, NY, 2008. Available at http://nyjm.albany.edu:8000/m/2008/3.htm.

[15] Iain Raeburn and Dana P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998.

[16] Jean Renault, *A groupoid approach to C^*-algebras*, Lecture Notes in Mathematics, vol. 793, Springer-Verlag, New York, 1980.

[17] ———, *Représentations des produits croisés d'algèbres de groupoïdes*, J. Operator Theory **18** (1987), 67–97.

[18] Marc A. Rieffel, *Induced representations of C^*-algebras*, Advances in Math. **13** (1974), 176–257.

[19] Dana P. Williams, *The topology on the primitive ideal space of transformation group C^*-algebras and C.C.R. transformation group C^*-algebras*, Trans. Amer. Math. Soc. **266** (1981), 335–359.

[20] ———, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence, RI, 2007.

[21] H. E. Winkelnkemper, *The graph of a foliation*, Ann. Global Anal. Geom. **1** (1983), 51–75.

Department of Mathematics, Cornell University, Ithaca, NY 14853-4201

Current address: Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009

E-mail address: `ionescu@math.uconn.edu`

Department of Mathematics, Dartmouth College, Hanover, NH 03755-3551

E-mail address: `dana.p.williams@Dartmouth.edu`