REMARKS ON THE IDEAL STRUCTURE OF
FELL BUNDLE C^*-ALGEBRAS

MARIUS IONESCU AND DANA P. WILLIAMS

ABSTRACT. We show that if $p : \mathscr{B} \to G$ is a Fell bundle over a locally compact groupoid G and that $A = \Gamma_0(G^{(0)}; \mathscr{B})$ is the C^*-algebra sitting over $G^{(0)}$, then there is a continuous G-action on $\operatorname{Prim} A$ that reduces to the usual action when $\mathscr{B}$ comes from a dynamical system. As an application, we show that if I is a G-invariant ideal in A, then there is a short exact sequence of C^*-algebras

$$0 \longrightarrow C^*(G, \mathscr{B}_I) \longrightarrow C^*(G, \mathscr{B}) \longrightarrow C^*(G, \mathscr{B}^I) \longrightarrow 0,$$

where $C^*(G, \mathscr{B})$ is the Fell bundle C^*-algebra and $\mathscr{B}_I$ and $\mathscr{B}^I$ are naturally defined Fell bundles corresponding to I and A/I, respectively. Of course this exact sequence reduces to the usual one for C^*-dynamical systems.

INTRODUCTION

An important source of examples of C^*-algebras are constructs associated to dynamics of some sort. Coming first on any such list would be group C^*-algebras followed closely by crossed product C^*-algebras coming from locally compact automorphism groups. More generally, we have the C^*-algebras associated to locally compact groupoids and groupoid dynamical systems. A very important extension of group dynamical systems is given by Fell bundles over groups. Fell bundles over groups were originally called (saturated) C^*-algebraic bundles by Fell, and are studied systematically in [13, Chap. VIII]. As illustrated by [8, Theorem 7.3], the C^*-algebra of a Fell bundle over a group G should be considered as a very general type of crossed product of the C^*-algebra sitting over the unit of G. Following Yamagami [27, 28], Kumjian formalized the notion of a Fell bundle $p : \mathscr{B} \to G$ over a locally compact groupoid in [21]. In this case, the associated C^*-algebra is meant to be a general type of crossed product of the C^*-algebra $A = \Gamma_0(G; \mathscr{B})$ of $\mathscr{B}$ sitting over the unit space $G^{(0)}$. This is illustrated by the examples in [23, §2].

The C^*-algebras of Fell bundles have been the object of considerable study starting with the group context [1, 7–11], then over étale groupoids [5, 14, 15] and eventually in the general setting [2–4, 20, 23]. This note is meant as a first step in a systematic investigation of the ideal structure of C^*-algebras associated to Fell bundles over locally compact (Hausdorff) groupoids. Our first result is to show that if $p : \mathscr{B} \to G$ is a Fell bundle and $A = \Gamma_0(G; \mathscr{B})$ is the C^*-algebra of $\mathscr{B}$ over $G^{(0)}$, then even though there is no explicit action of G on A, there is a natural G-action of G on the primitive ideal space $\operatorname{Prim} A$ of A which generalizes the usual notion when $\mathscr{B}$ is the Fell bundle associated to either a classical C^*-dynamical system or a groupoid dynamical system.

Once we have a G-action on $\operatorname{Prim} A$, then it makes sense to speak of an invariant ideal I of A. Our next contribution is that to each G-invariant ideal I in A, there are naturally associated Fell bundles $p_I : \mathscr{B}_I \to G$ and $p^I : \mathscr{B}^I \to G$ corresponding to I and A/I, respectively, and a short exact sequence

$$0 \longrightarrow C^*(G, \mathscr{B}_I) \longrightarrow C^*(G, \mathscr{B}) \longrightarrow C^*(G, \mathscr{B}^I) \longrightarrow 0$$

of C^*-algebras. This result generalizes the fundamental result in C^*-dynamical systems that asserts that if (A, G, α) is a C^*-dynamical system with I an α-invariant ideal in A, then there is a short exact sequence

$$0 \longrightarrow I \rtimes_\alpha G \longrightarrow A \rtimes_\alpha G \longrightarrow (A/I) \rtimes_{\alpha^I} G \longrightarrow 0.$$

(For example, see [26, Proposition 3.19].) However, the proof is considerably more subtle in our setting and requires the Disintegration Theorem for Fell bundles [26, Theorem 4.13].

In subsequent work, we intend to use these elementary results together with [19] to study the Mackey machine for Fell bundles and the fine ideal structure of the corresponding C^*-algebras.

We adopt the usual conventions in the subject. Representations of C^*-algebras will be assumed to be nondegenerate, and homomorphisms between C^*-algebras will always be $*$-preserving. We will write $M(A)$ for the multiplier algebra of a C^*-algebra A. As in [22, 25], we view $M(A)$ as the set of adjointable operators $\mathcal{L}(A)$ where A is viewed as a Hilbert module over itself. We write $\tilde{A}$ for the C^*-subalgebra of $M(A)$ generated by A and 1_A. Thus $\tilde{A}$ is simply A if A has a unit and A with a unit adjoined otherwise.

1. Preliminaries

A Fell bundle $p : \mathscr{B} \to G$ over a locally compact Hausdorff groupoid G is an upper semicontinuous Banach bundle equipped with a continuous, bilinear, associative multiplication map $(a, b) \mapsto ab$ from $\mathscr{B}^{(2)} := \{ (a, b) \in \mathscr{B} \times \mathscr{B} : (p(a), p(b)) \in G^{(2)} \}$ to $\mathscr{B}$ and an involution $b \mapsto b^*$ from $\mathscr{B}$ to $\mathscr{B}$ satisfying axioms (a)–(e) of [23, Definition 1.1]. We will adopt the notations and conventions of [23] and refer to [23, §1] for the construction of the associated C^*-algebra $C^*(G, \mathscr{B})$ built from the $*$-algebra of continuous compactly supported sections $\Gamma_c(G; \mathscr{B})$ of $\mathscr{B}$. In particular, our Fell bundles are *saturated* in that $B(x)B(y) := \overline{\operatorname{span}}\{ ab : a \in B(x) \text{ and } b \in B(y) \} = B(xy)$. Since we make considerable use of the Disintegration Theorem for Fell bundles (see [23, Theorem 4.13]), we will need to assume that all our Fell bundles are *separable* in that G is second countable and that the Banach space $\Gamma_0(G; \mathscr{B})$ is separable. It is important to keep in mind that the property that each fibre $B(x)$ is an imprimitivity bimodule has important consequences. For example, it follows that $b^*b \geq 0$ in the C^*-algebra sitting over $s(b) := s(p(b))$, and that $\|b^*b\| = \|b\|^2$ for all $b \in \mathscr{B}$. Moreover, we have the following useful observation.

Lemma 1. *If $p : \mathscr{B} \to G$ is a Fell bundle then $\|ab\| \leq \|a\|\|b\|$ for all $(a, b) \in \mathscr{B}^{(2)}$.*

Proof. The lemma follows from [23, Lemma 1.2] once we prove the following observation (which is probably known to specialists but we have been unable to find a reference in the literature): If A, B, C are C^*-algebras, and $_A X_B$ and $_B Y_C$ are

imprimitivity bimodules,[1] then

$$\|x \otimes y\| \le \|x\|\|y\|$$

for all $x \otimes y \in X \otimes_B Y$. The proof of this observation follows from the computation:

$$\|x \otimes y\|^2 = \|\langle x \otimes y \ , \ x \otimes y \rangle_C\| = \big\|\big\langle \langle x \ , \ x \rangle_B \cdot y \ , \ y \big\rangle_C\big\|$$
$$= \big\|\big\langle \langle x \ , \ x \rangle_B^{1/2} \cdot y \ , \ \langle x \ , \ x \rangle_B^{1/2} \cdot y \big\rangle_C\big\|.$$

If T is an adjointable operator on Y we know that $\langle Ty \ , \ Ty \rangle_C \le \|T\|^2\|\langle y \ , \ y \rangle_C\|$. Since B acts on the left on Y via adjointable operators it follows that

$$\|x \otimes y\|^2 \le \big\|\|\langle x \ , \ x \rangle_B^{1/2}\|^2 \langle y \ , \ y \rangle_C\big\| = \|x\|^2\|y\|^2. \qquad \square$$

We will also require a number of basics concerning upper semicontinuous Banach bundles. The definition of an upper semicontinuous-Banach bundle, as well as a collection of basic results and original sources, are given in [23, Appendix A]. Actually, much of what is needed is covered in detail in [26, Appendix C.2]; unfortunately, all the results in [26, Appendix C.2] are stated in terms of upper semicontinuous C^*-bundles even though the additional C^*-structure is not always necessary for the result. For example, the proof of [26, Proposition C.20], which characterizes convergence in the total space, makes no use of the C^*-axioms and remains valid for upper semicontinuous-Banach bundles by simply replacing "upper semicontinuous C^*-bundle over X" by "upper semicontinuous-Banach bundle over X". Therefore we will sheepishly, but firmly, cite results like [26, Proposition C.20] below for upper semicontinuous-Banach bundles with the implicit understanding that the appropriate Banach bundle result is valid with essentially the same proof as given in [26, Appendix C.2]. For example, if $p : \mathscr{B} \to X$ is an upper semicontinuous-Banach bundle over X, then we note that $\Gamma_0(X; \mathscr{B})$ is a Banach space in the sup-norm by (the same proof as given in) [26, Proposition C.23].

Although the total space of a Banach bundle is an important theoretical tool in groupoid constructs, in the wild Banach bundles are built by first specifying what the sections should be. As far as we know, the next result is originally due to Hofmann [17, 18], although only the details for continuous bundles were ever published (see, for example, [12, Theorem II.13.18]).[2] What is needed is the following.

Theorem 2 (Hofmann-Fell). *Let X be a locally compact space and suppose that for each $x \in X$, we are given a Banach space $B(x)$. Let $\mathscr{B}$ be the disjoint union $\coprod_{x \in X} B(x)$, and let $p : \mathscr{B} \to X$ be the obvious bundle. Suppose that Γ is a subspace of sections such that*

    (a) *for each $f \in \Gamma$, $x \mapsto \|f(x)\|$ is upper semicontinuous, and*
    (b) *for each $x \in X$, $\{\, f(x) : f \in \Gamma \,\}$ is dense in $B(x)$.*

---

[1]All that is required here is for X and Y to be a right Hilbert modules with actions of A and B, respectively, coming from homomorphisms into the adjointable operators. Such objects are either called C^*-*correspondences* or *right Hilbert bimodules* in the literature.

[2]Recently, Buss and Exel have published a version of this result which does not even require the base space X to be Hausdorff: [4, Proposition 2.4]. In their result, they make the extra assumption that $\{\, f(x) : f \in \Gamma \,\}$ is all of $B(x)$. This is required to ensure that the topology is unique when X fails to be Hausdorff. In our applications, this extra assumption on the fibres is satisfied, so we could also appeal to [4, Proposition 2.4].

Then there is a unique topology on $\mathscr{B}$ such that $p : \mathscr{B} \to X$ is an upper semicontinuous-Banach bundle over X with $\Gamma \subset \Gamma(X; \mathscr{B})$.

Proof. This result is stated in [6, Proposition 1.3] with a reference to [17]. With the proviso discussed above, it follows from [26, Theorem C.25]. $\qquad\square$

Remark 3. If $p : \mathscr{B} \to G$ is a Fell bundle over a *group* G, then the underlying Banach bundle is necessarily continuous. This was observed in [2, Lemma 3.30] and is apparently due to Exel. The idea is that $a \mapsto (a^*, a) \mapsto a^* a \mapsto \|a^* a\|^{\frac{1}{2}}$ must be continuous as maps $\mathscr{B} \to \mathscr{B} \times \mathscr{B} \to A \to [0, \infty)$. The same argument shows that if $p : \mathscr{B} \to G$ is a Fell bundle over a locally compact Hausdorff groupoid G, then the underlying Banach bundle is continuous if and only if the associated C^*-algebra $A := \Gamma_0(G^{(0)}; \mathscr{B})$ is a continuous field over $G^{(0)}$.

If $p : \mathscr{B} \to G$ is a Fell bundle over a locally compact Hausdorff groupoid G, then

$$A := \Gamma_0(G^{(0)}; \mathscr{B})$$

is a C^*-algebra which we call *the C^*-algebra of $\mathscr{B}$ sitting over $G^{(0)}$*. Note that A is a $C_0(G^{(0)})$-algebra, and let $\sigma : \operatorname{Prim} A \to G^{(0)}$ be the associated structure map.[3] If $u \in G^{(0)}$, let $q_u : A \to A(u)$ be the quotient map with kernel I_u. Then $\sigma(P) = u$ if and only if $I_u \subset P$.[4]

Furthermore, $\operatorname{Prim} A$ is naturally identified with the disjoint union of the $\operatorname{Prim} A(u)$ [26, Proposition C.5]. Thus we will write

$$\operatorname{Prim} A = \{\, (u, P) : u \in G^{(0)} \text{ and } P \in \operatorname{Prim} A(u) \,\}.$$

It will be important to keep in mind that $(u, P) = q_u^{-1}(P)$.

We will need the following technical lemma on $C_0(X)$-algebras in the proof of Proposition 15.

Lemma 4. *Suppose that A is a $C_0(X)$-algebra with structure map $\sigma : \operatorname{Prim} A \to X$ (see [26, Proposition C.5]), and let $A(x) = A/I_x$ be the fibre over x. Let K be an ideal in A, and let*

$$F = \{\, P \in \operatorname{Prim} A : P \supset K \,\}$$

be the closed subset of $\operatorname{Prim} A$ identified with $\operatorname{Prim}(A/K)$. Then $\sigma|_F$ induces a $C_0(X)$-structure on A/K and $(A/K)(x) = (A/K)/\mathscr{I}_x$, where $\mathscr{I}_x = (I_x + K)/K \cong I_x/(I_x \cap K)$. In particular, $(A/K)(x) \cong A/(K + I_x)$.

Proof. Recall that the Dauns-Hofmann Theorem [26, Theorem A.24] gives an isomorphism Φ_A of $C^b(\operatorname{Prim} A)$ onto the center $ZM(A)$ of the multiplier algebra which is characterized by

$$(\Phi_A(f)a)(P) = f(P)a(P),$$

where $a(P)$ denotes the image of a in A/P. Then we view A as a $C_0(X)$-module via

$$\varphi \cdot a = \Phi_A(\varphi \circ \sigma)a.$$

Also recall that

$$I_x = \overline{\operatorname{span}}\{\, \varphi \cdot a : a \in A \text{ and } \varphi \in J_x \,\},$$

---

[3] For a summary of basic results and our notations for $C_0(X)$-algebras, please refer to [26, §C.1].

[4] For $u \in G^{(0)}$, the fibres $A(u)$ and $B(u)$ are identical as sets. If there is an excuse for using different letters, it is that $A(u)$ is meant to be thought of as a C^*-algebra and $B(u)$ is $A(u)$ viewed as a $A(u) - A(u)$-imprimitivity bimodule.

where $J_x = \{\, \varphi \in C_0(X) : \varphi(x) = 0 \,\}$. Similarly, if $q : A \to A/K$ is the quotient map, then

$$\mathscr{I}_x = \overline{\mathrm{span}}\{\, \varphi \cdot q(a) : a \in A \text{ and } \varphi \in J_x \,\}.$$

Notice that if $P \supset K$, then

$$\varphi \cdot q(a)(P/K) = \varphi(\sigma(P))q(a)(P/K) = q(\varphi(\sigma(P))a)(P/K).$$

On the other hand, if $P \supset K$, then the natural isomorphism of $(A/K)/(P/K)$ with A/P carries $q(a)(P/K)$ to $a(P)$. It follows that

$$q(\varphi(\sigma(P))a)(P/K) = q(\varphi \cdot a)(P/K).$$

Thus,

$$\mathscr{I}_x = \overline{\mathrm{span}}\{\, q(\varphi \cdot a) : a \in A \text{ and } \varphi \in J_x \,\} = q(I_x) = (I_x + K)/K.$$

For the final statement, notice that

$$(A/K)(x) = (A/K)/\mathscr{I}_x = (A/K)/((I_x + K)/K) \cong A/(I_x + K). \qquad \square$$

We will make use of the following remark in §3.

Remark 5. Note that if X is a $A - B$-imprimitivity bimodule and if J is an ideal in A, then $Y := \overline{\mathrm{span}}\{\, a \cdot x : a \in J \text{ and } x \in X \,\}$ is a nondegenerate J-module. Then, employing the Cohen Factorization Theorem ([25, Proposition 2.33]),

$$Y = \{\, a \cdot y : a \in J \text{ and } y \in Y \,\} \subset \{\, a \cdot x : a \in J \text{ and } x \in X \,\}$$

$$\subset \overline{\mathrm{span}}\{\, a \cdot x : a \in J \text{ and } x \in X \,\} = Y.$$

Consequently, $Y = \{\, a \cdot x : a \in J \text{ and } x \in X \,\}$, and will routinely write $J \cdot X$ in place of Y. Similarly, we'll write $X \cdot I$ for the corresponding $A - B$-submodule when I is an ideal in B.

2. The G-action on $\mathrm{Prim}\, A$

Now we want to see that $\mathrm{Prim}\, A$ admits a G-action. The key is to recall that for each $x \in G$, $B(x)$ is a $A\big(r(x)\big) - A\big(s(x)\big)$-imprimitivity bimodule. Thus by [25, Corollary 3.33], the *Rieffel correspondence* defines a homeomorphism

$$h_x : \mathrm{Prim}\, A\big(s(x)\big) \to \mathrm{Prim}\, A\big(r(x)\big)$$

where h_x is the restriction of $B(x)$–Ind to $\mathrm{Prim}\, A\big(s(x)\big)$.[5] We will use the convenient facts that $h_{x^{-1}}$ is the inverse to h_x, and that h_x is containment preserving [25, Corollary 3.31].

Then we can define

(1) $$x \cdot \big(s(x), P\big) := \big(r(x), h_x(P)\big).$$

Suppose that $(x, y) \in G^{(2)}$ and that L is a representation of $A\big(s(x)\big)$. It is not hard to check that

$$B(x)\text{–Ind}\big(B(y)\text{–Ind}(L)\big) \cong \big(B(x) \otimes_{A(s(x))} B(y)\big)\text{–Ind}(L).$$

Since $B(x) \otimes_{A(s(x))} B(y)$ and $B(xy)$ are isomorphic as imprimitivity bimodules by [23, Lemma 1.2], it follows from [25, Proposition 3.24] that

$$h_x \circ h_y = h_{xy}.$$

---

[5]Recall that if X is an $A - B$-imprimitivity bimodule, then X–Ind is a continuous map from $\mathcal{I}(B)$ to $\mathcal{I}(A)$ characterized by X–Ind$(\ker L) = \ker\big(X\text{–Ind}(L)\big)$ for representations L of B (see [25, Proposition 3.24]).

Since $h_u = \mathrm{id}_{\mathrm{Prim}\,A(u)}$, it follows that (1) defines an action of G on $\mathrm{Prim}\,A$.

Lemma 6. *Suppose that $x \in G$ and that $P \in \mathrm{Prim}\,A\big(s(x)\big)$. Then*

$$h_x(P) = \overline{\mathrm{span}}\{\, adb^* : a, b \in B(x) \text{ and } d \in P \,\}.$$

Proof. By the axioms for Fell bundles, the $A\big(r(x)\big)$-valued inner product on the imprimitivity bimodule $B(x)$ is given by $\langle a \,,\, b \rangle_* = ab^*$. Then we can apply [25, Proposition 3.24] to check that

$$
\begin{aligned}
h_x(P) &:= B(x)\text{–}\mathrm{Ind}(P) \\
&= \overline{\mathrm{span}}\{\, {}_*\langle ad \,,\, b \rangle : a, b \in B(x) \text{ and } d \in P \,\} \\
&= \overline{\mathrm{span}}\{\, adb^* : a, b \in B(x) \text{ and } d \in P \,\} \qquad \square
\end{aligned}
$$

Remark 7. We want to see that the G-action given by (1) is the same as the usual one when $p : \mathscr{B} \to G$ is the Fell bundle associated to a dynamical system (D, G, α) (as in [23, Example 2.1]). To start with, assume that G is a group. In this case $\mathscr{B}$ is the trivial bundle $D \times G$ and the multiplication in $\mathscr{B}$ is given by $(a, s)(b, t) = (a\alpha_s(g), st)$. Although it is tempting to simply identify D with $A(e)$, it is useful to distinguish the two. In particular, if $P \in \mathrm{Prim}\,A(e)$, then there is an ideal $\overline{P} \in \mathrm{Prim}\,D$ such that $P = \{\,(d, e) : d \in \overline{P}\,\}$. Then Lemma 6 implies that

$$
\begin{aligned}
h_s(P) &= \overline{\mathrm{span}}\{\, (a, s)(d, e)(\alpha_{s^{-1}}(b^*), s^{-1}) : a, b \in D \text{ and } d \in \overline{P} \,\} \\
&= \overline{\mathrm{span}}\{\, (a\alpha_s(d)b^*, e) : a, b \in D \text{ and } d \in \overline{P} \,\} \\
&= (\alpha_s(\overline{P}), e),
\end{aligned}
$$

and we recover the usual thing.

If G is a groupoid and $D = \Gamma_0(G^{(0)}; \mathscr{D})$, then $\mathscr{B}$ is the pull-back $r^*\mathscr{D}$, and the multiplication in $\mathscr{B}$ is given by

$$(a, x)(b, y) = \big(a\alpha_x(b), xy\big).$$

Again, it is useful to distinguish the fibre $D(u)$ and its image $A(u) = \{\,(d, u) : d \in D(u)\,\}$ in $\mathscr{B}$. Now, invoking Lemma 6, if $P \in \mathrm{Prim}\,A\big(s(x)\big)$ and $\overline{P}$ is the ideal in $D\big(s(x)\big)$ such that $P = \{\,(d, s(x)) : d \in \overline{P}\,\}$, then

$$
\begin{aligned}
h_x(P) &= \overline{\mathrm{span}}\{\, (a, x)(d, s(x))(\alpha_{x^{-1}}(b^*), x^{-1}) : a, b \in A\big(r(x)\big) \text{ and } d \in \overline{P} \,\} \\
&= \overline{\mathrm{span}}\{\, \big(a\alpha_x(d)b^*, r(x)\big) : a, b \in A\big(r(x)\big) \text{ and } d \in \overline{P} \,\} \\
&= \big(\alpha_x(\overline{P}), r(x)\big) \\
&= \alpha_x(P).
\end{aligned}
$$

Therefore we recover the usual G-action on $\mathrm{Prim}\,A$ in this case as well.

Remark 8 (Viewing h_x as a map on ideals). To ease the notational burden, we will also write h_x for the map $B(x)\text{–}\mathrm{Ind} : \mathcal{I}\big(A(s(x))\big) \to \mathcal{I}\big(A(r(x))\big)$. In view of [25, Theorem 3.29], it is still the case that $h_{x^{-1}}$ is the inverse to h_x, and of course, h_x is still containment preserving. Note that if J is an ideal in A, then its image $q_u(J)$ in $A(u)$ is $\{\, c(u) : c \in J \,\}$. In particular, repeating the proof of Lemma 6, we see that

$$(2) \qquad h_x\big(q_{s(x)}(J)\big) = \overline{\mathrm{span}}\{\, ac\big(s(x)\big)b^* : a, b \in B(x) \text{ and } c \in J \,\}.$$

Proposition 9. *If $p : \mathscr{B} \to G$ is a Fell bundle and A is the associated C^*-algebra over $G^{(0)}$, then the G-action on $\operatorname{Prim} A$ defined by (1) is continuous and $\operatorname{Prim} A$ is a G-space.*

Proof. At this point, we just need to show that if $(x_i, \overline{P}_i) \to (x_0, \overline{P}_0)$ in $G * \operatorname{Prim} A$, then $x_i \cdot \overline{P}_i \to x_0 \cdot \overline{P}_0$ in $\operatorname{Prim} A$. For convenience, let $\overline{P}_i = (s(x_i), P_i)$.

Suppose that $x_0 \cdot \overline{P}_0 \in \mathcal{O}_J = \{ K \in \operatorname{Prim} A : K \not\supset J \}$. Then it will suffice to see that $x_i \cdot \overline{P}_i$ is eventually in $\mathcal{O}_J$. Suppose not. Then we can pass to a subnet, relabel, and assume that for all $i \neq 0$, we have

$$(3) \qquad x_i \cdot (s(x_i), P_i) \supset J.$$

Since $x_i \cdot (s(x_i), P_i) = (r(x_i), h_{x_i}(P_i))$, (3) implies that $h_{x_i}(P_i) \supset q_{r(x_i)}(J)$. Since the inverse of h_{x_i} is $h_{x_i^{-1}}$, we have

$$(4) \qquad P_i \supset h_{x_i^{-1}}\big(q_{r(x_i)}(J)\big) \quad \text{for all } i \neq 0.$$

We claim it will suffice to see that (4) holds for $i = 0$. To see this, notice that if (4) holds for $i = 0$, then

$$h_{x_0}(P_0) \supset q_{r(x_0)}(J),$$

and since $(r(x), h_x(P)) = q_{r(x)}^{-1}\big(h_x(P)\big)$, this implies

$$x_0 \cdot (s(x_0), P_0) = (r(x_0), h_{x_0}(P_0)) \supset q_{r(x_0)}^{-1}\big(q_{r(x_0)}(J)\big) \supset J.$$

But this contradicts the assumption that $x_0 \cdot \overline{P}_0 \in \mathcal{O}_J$ and will complete the proof.

To establish the claim, we notice that when $i = 0$ the right-hand side of (4) is given by

$$\overline{\operatorname{span}}\{ a^* c\big(r(x_0)\big)b : a, b \in B(x) \text{ and } c \in J \},$$

where we have invoked the fact that $B(x^{-1}) = B(x)^*$ and used (2) from Remark 8. Thus it will suffice to show that for any $c \in J$ and $a, b \in B(x_0)$, we have $a^* c\big(r(x_0)\big)b \in P_0$.

Since we always assume that Fell bundles have enough sections,[6] there are $f, g \in \Gamma_c(G; \mathscr{B})$ such that $f(x_0) = a$ and $g(x_0) = b$. Then we can form a section $\xi \in \Gamma_c(G; r^*\mathscr{A})$ in the C^*-algebra $C := \Gamma_0(G; r^*\mathscr{A})$ given by

$$\xi(x) := f(x)^* c\big(r(x)\big)g(x).$$

Notice that if π is an irreducible representation of A with $\sigma(\ker \pi) = u$, then there is an associated irreducible representation $\bar{\pi}$ of $A(u)$ such that $\pi = \bar{\pi} \circ q_u$. If $x \in G$ and $r(x) = u$, then we get an irreducible representation $[x, \pi]$ of C by $[x, \pi](f) = \bar{\pi}\big(f(x)\big)$. Furthermore, by [24, Proposition 1.3 and Lemma 1.2], the spectrum of C is homeomorphic to

$$\{ [x, \pi] \in G \times \hat{A} : r(x) = \sigma(\ker \pi) \}.$$

Now let π_i be an irreducible representation of A with kernel $\overline{P}_i$. Then, since the topology on $\hat{A}$ is pulled back from the topology on $\operatorname{Prim} A$, $\pi_i \to \pi_0$ in $\hat{A}$. Consequently,

$$(5) \qquad [x_i^{-1}, \pi_i] \to [x_0^{-1}, \pi_0] \quad \text{in } \hat{C}.$$

By construction,

$$\xi(x_i) \in \overline{\operatorname{span}}\{ a^* db : a, b \in B(x_i) \text{ and } d \in q_{r(x_i)}(J) \} = h_{x_i^{-1}}\big(q_{r(x_i)}(J)\big).$$

---

[6]This is actually automatic. See the comments on page 51 of [23, Appendix A].

Hence $\xi(x_i) \in P_i = \ker \bar{\pi}_i$ by (4). Therefore

$$[x_i^{-1}, \pi_i](\xi) = 0 \quad \text{for all } i \neq 0.$$

Therefore $[x_0^{-1}, \pi_0](\xi) = 0$ by (5). Since c was an arbitrary element of J, this proves that (4) holds for $i = 0$ and completes the proof. $\qquad\square$

3. Invariant Ideals

It is a classic result in crossed products ([16, Proposition 12]) that if (A, G, α) is a dynamical system and if I is an α-invariant ideal, then there is a short exact sequence

$$0 \longrightarrow I \rtimes_\alpha G \xrightarrow{\iota \rtimes \mathrm{id}} A \rtimes_\alpha G \xrightarrow{q \rtimes \mathrm{id}} (A/I) \rtimes_{\alpha^I} G \longrightarrow 0$$

(see [26, Proposition 3.19]). In this section, we want to prove a similar result for Fell bundles. This entails some nontrivial work. Even to start, we need to determine what an invariant ideal is, and which C^*-algebras correspond to I and the quotient A/I.

3.1. **Preliminaries.** We assume that $p : \mathscr{B} \to G$ is a separable Fell bundle over a locally compact Hausdorff groupoid G. Let $A = \Gamma_0(G^{(0)}; \mathscr{B})$ be the C^*-algebra over $G^{(0)}$. We say that an ideal I in A is G-invariant if the closed set

$$\mathrm{hull}(I) = \{\, \overline{P} \in \mathrm{Prim}\, A : \overline{P} \supset I \,\}$$

is a G-invariant subset of $\mathrm{Prim}\, A$ with respect to the G-action introduced in Proposition 9.

Now fix an ideal $I \in \mathcal{I}(A)$. If $q_u : A \to A(u) = A/J_u$ is the quotient map, then we let $I(u) := q_u(I) = (I + J_u)/J_u$.

Lemma 10. *Let $h_x : \mathcal{I}\big(A(s(x))\big) \to \mathcal{I}\big(A(r(x))\big)$ be the Rieffel correspondence. If I is a G-invariant ideal, then*

$$h_x\big(I(s(x))\big) = I(r(x)).$$

In particular, $B(x) \cdot I(s(x)) = I(r(x)) \cdot B(x)$.

Remark 11. We have included a `\cdot` in the above notation to stress that $B(x) \cdot I(s(x))$ is the sub-bimodule corresponding to $I(s(x))$ in the imprimitivity bimodule $B(x)$. Since the right action is just given by multiplication in $\mathscr{B}$, the `\cdot` can be dropped without any harm. In fact, it will be critical in what follows that we are just dealing with multiplication in $\mathscr{B}$ which is an associative operation (when that makes sense).

Proof. If $P \in \mathrm{Prim}\big(A(s(x))\big)$ and I is invariant, then

$$\begin{aligned}
P \supset I(s(x)) &\iff \big(s(x), P\big) \supset I \\
&\iff \big(r(x), h_x(P)\big) \supset I \\
&\iff h_x(P) \supset I(r(x)).
\end{aligned}$$

The first assertion follows. The second assertion follows from [25, Proposition 3.24]. $\qquad\square$

Now we define

$$\mathscr{B}_I := \{\, b \in \mathscr{B} : b^*b \in I(s(b)) \,\},$$

where, as is usual, we write $s(b)$ as a shorthand for $s(p(b))$.

Lemma 12. *If I is an ideal in A, then*

$$\mathscr{B}_I = \{\, b \in \mathscr{B} : a^* b \in I(s(b)) \text{ for all } a \in B(p(b)) \,\}.$$

In particular, $b \in \mathscr{B}_I$ implies that $b \in B(p(b)) \cdot I(s(b))$.

Remark 13. Note that $B(x) \cdot I(s(b))$ is always an imprimitivity bimodule between $I(s(b))$ and the ideal of $A(r(x))$ corresponding to $I(s(b))$ under the Rieffel correspondence [25, Proposition 3.25].

Proof. Since $B(p(b))$ is a right Hilbert $A(s(b))$-module, this result follows from [25, Lemma 3.23]. $\qquad\square$

Proposition 14. *Suppose that $p : \mathscr{B} \to G$ is a Fell bundle over a locally compact Hausdorff groupoid G and that I is an ideal in the C^*-algebra $A = \Gamma_0(G^{(0)}; \mathscr{B})$. Let $\mathscr{B}_I = \{\, b \in \mathscr{B} : b^* b \in I(s(b)) \,\}$ and $p_I = p|_{\mathscr{B}_I}$. Then $p_I : \mathscr{B}_I \to G$ is an upper semicontinuous-Banach bundle with fibres $\mathscr{B}_I(x) = B(x) \cdot I(s(b))$.[7] If I is G-invariant, then $\mathscr{B}_I$ is a Fell bundle with the operations inherited from $\mathscr{B}$.*

Proof. It is fairly straightforward to check that $p_I : \mathscr{B}_I \to G$ is an upper semicontinuous-Banach bundle with the exception of verifying that p_I is open.[8] To prove that, we'll use [26, Proposition 1.15]. Suppose that $b \in \mathscr{B}_I$ and that $x_i \to p(b)$. It will suffice to find, after passing to a subsequence and relabeling, elements $b_i \in \mathscr{B}_I$ such that $b_i \to b$ and $p(b_i) = x_i$.

Since $\mathscr{B}_I(p(b)) = B(p(b)) \cdot I(s(b))$, in view of Remark 5 we can suppose that $b = b' \cdot a(s(b))$ where $a \in I$ (and we write $a(u)$ for the image of a in $I(u)$). Since p is open, we can pass to a subsequence, relabel, and find $b'_i \to b'$ such that $p(b'_i) = x_i$. But $a(s(x_i)) \to a(s(b))$ in $\mathscr{B}$, so the continuity of multiplication implies that $b'_i \cdot a(s(x_i)) \to b' \cdot a(s(b)) = b$. But $b_i := b'_i \cdot a(s(x_i)) \in \mathscr{B}_I$. This completes the proof that p_I is open and the proof that p_I is an upper semicontinuous-Banach bundle.

Now we assume that I is G-invariant. Suppose that $(b, b') \in \mathscr{B}_I^{(2)}$. Then $b \in B(p(b)) \cdot I(s(b))$, $b' \in B(p(b')) \cdot I(s(b'))$ and $s(b) = r(b')$. Thus using Lemma 10,

$$
\begin{aligned}
bb' &\in B(p(b)) I(s(b)) B(p(b')) I(s(b')) \\
&\subset B(p(b)) B(p(b')) I(s(b'))^2 \\
&\subset B(p(bb')) I(s(bb')).
\end{aligned}
$$

Thus $bb' \in \mathscr{B}_I$ and $\mathscr{B}_I$ is closed under multiplication. Similarly, $b^* \in I(s(b)) B(p(b)^{-1}) = B(p(b^*)) I(s(b^*))$, and $b^* \in \mathscr{B}_I$.

Therefore axioms (a)–(c) of [23, Definition 1.1] are clearly satisfied. Since $\mathscr{B}_I(u) = I(u)$, axiom (d) is satisfied, and [25, Proposition 3.25] together with Lemma 10 imply that $\mathscr{B}_I(x)$ is an $I(r(x)) - I(s(x))$-imprimitivity bimodule. Thus (e) holds as well. $\qquad\square$

Now we let $B^I(x)$ be the quotient module $B(x)/B_I(x)$. If J is the ideal of $A(r(x))$ corresponding to $I(s(x))$ under the Rieffel correspondence (so that $J = I(r(x))$ if I is invariant by Lemma 10), then $B^I(x)$ is a $A(r(x))/J - A(s(x))/I(s(x))$-imprimitivity bimodule by [26, Proposition 3.25]. Let

$$\mathscr{B}^I := \coprod_{x \in G} B^I(x)$$

-----

[7]It is possible — even likely — that some of the fibres $\mathscr{B}_I(x)$ are the zero space.

[8]Keep in mind that the restriction of an open map need not be open.

and form the bundle $p^I : \mathscr{B}^I \to G$. If $f \in \Gamma_c(G; \mathscr{B})$, then we define a section $q(f)$ of $p^I : \mathscr{B}^I \to G$ by $q(f)(x) = [f(x)]$, where as usual, $[b]$ denotes the class of $b \in B(x)$ in $B^I(x)$.

Proposition 15. *Suppose that $p : \mathscr{B} \to G$ is a Fell bundle over a locally compact Hausdorff groupoid G and that I is an ideal in the C^*-algebra $A = \Gamma_0(G^{(0)}; \mathscr{B})$. Then $\mathscr{B}^I$ has a topology making $p^I : \mathscr{B}^I \to G$ an upper semicontinuous-Banach bundle such that $\Gamma := \{ q(f) : f \in \Gamma_c(G; \mathscr{B}) \}$ is a dense subspace of $\Gamma_c(G; \mathscr{B}^I)$ in the inductive limit topology. If I is G-invariant, then $p^I : \mathscr{B}^I \to G$ is a Fell bundle with the operations induced from $\mathscr{B}$.*

Proof of Proposition 15. Recall that $B^I(x)$ is a quotient imprimitivity module, which is in particular a right Hilbert $A(s(x))/I(s(x))$-module, where the Hilbert module operations are induced from the Fell bundle operations on $\mathscr{B}$. Therefore,

$$\|q(f)(x)\|^2 = \|\bar{q}(f(x)^*f(x))\|,$$

where $\bar{q} : A \to A/I(s(x))$ is the quotient map. But Lemma 4 implies that A/I is a $C_0(G^{(0)})$-algebra with fibre over u, $(A/I)(u)$, naturally isomorphic to $A(u)/I(u)$. Therefore,

$$\|q(f)(x)\|^2 = \|q_I(f(x)^*f(x))(s(x))\|,$$

where $q_I : A \to A/I$ is the quotient map. Therefore, $x \mapsto \|q(f)(x)\|$ is the composition of the continuous map $x \mapsto f(x)^*f(x)$ from G to A with the upper semicontinuous map $q_I(a) \mapsto \|q_I(a)(u)\|$ coming from the $C_0(G^{(0)})$-algebra structure ([26, Proposition C.10]). Therefore $x \mapsto \|q(f)(x)\|$ is upper semicontinuous and we can apply Theorem 2 to give $\mathscr{B}^I$ a topology such that $p^I : \mathscr{B}^I \to G$ is an upper semicontinuous-Banach bundle such that $\Gamma := \{ q(f) : g \in \Gamma_c(G; \mathscr{B}) \} \subset \Gamma_c(G; \mathscr{B}^I)$. Furthermore, since Γ is a $C_0(G)$-module, it follows from [23, Lemma A.4] that Γ is dense in $\Gamma_c(G; \mathscr{B}^I)$ in the inductive limit topology. This establishes all but the last assertion.

Now assume that I is G-invariant. Let $(x, y) \in G^{(2)}$. Suppose that $a, a' \in B(x)$, that $b, b' \in B(y)$ and that $a' = a + a''$, $b' = b + b''$ with $a'' \in B_I(x)$ and $b'' \in B_I(y)$. Since $s(x) = r(y)$, we can apply Lemma 10 to calculate as follows:

$$a'b' = ab + a''b + ab'' + a''b''$$
$$\in ab + B(x)I(r(y))B(y) + B(x)B(y)I(s(y)) + B(x)I(r(y))B(y)I(s(y))$$
$$\subset ab + B(xy)I(s(y)).$$

Therefore, we get a well-defined multiplication on $(\mathscr{B}^I)^{(2)}$.

To see that multiplication is continuous from $(\mathscr{B}^I)^{(2)}$ to $\mathscr{B}^I$ we first want to establish that the quotient map $a \mapsto [a]$ is continuous from $\mathscr{B}$ to $\mathscr{B}^I$. To this end, suppose that $a_i \to a$ in $\mathscr{B}$. Let $f \in \Gamma_c(G; \mathscr{B})$ be such that $f(p(a)) = a$. Then

$$\|f(p(a_i)) - a_i\| \to 0.$$

But then

$$\|q(f)(p(a_i)) - [a_i]\| \to 0.$$

Since $q(f) \in \Gamma_c(G; \mathscr{B}^I)$, we have $[a_i] \to [a]$ by [26, Proposition C.20].

Now suppose that $([a_i], [b_i]) \to ([a], [b])$ in $(\mathscr{B}^I)^{(2)}$ with $\big(p^I([a_i]), p^I([b_i])\big) = (x_i, y_i)$ converging to (x, y) in $G^{(2)}$. Let $f, g \in \Gamma_c(G; \mathscr{B})$ be such that $q(f)(x) = [a]$ and $q(g)(x) = [b]$. By the above observation, we have

$$q(f)(x_i)q(g)(y_i) \to [a][b].$$

However, using Lemma 1, we see easily that

$$\|q(f)(x_i)q(g)(y_i) - [a_i][b_i]\| \to 0.$$

Therefore $[a_i][b_i] \to [a][b]$ by [26, Proposition C.20], and multiplication is continuous. The argument that $[a] \mapsto [a^*]$ is a well defined and continuous involution is similar and we omit the details.

It is now straightforward to see that $p^I : \mathscr{B}^I \to G$ is a Fell bundle as claimed: properties (a)–(c) of [23, Definition 1.1] are clearly satisfied. On the other hand, if $u \in G^{(0)}$, then $B^I(u)$ is the C^*-algebra $A(u)/I(u) = (A/I)(u)$, while if $x \in G$, then $B^I(x)$ is a $(A/I)(r(x)) - (A/I)(s(x))$-imprimitivity bimodule with respect to the quotient operations. Therefore properties (d) and (e) are also satisfied. $\qquad\square$

3.2. **The Exact Sequence.** In this section, we assume throughout that I is a G-*invariant ideal* in the C^*-algebra $A = \Gamma_0(G^{(0)}; \mathscr{B})$ sitting over $G^{(0)}$ in a Fell bundle $p : \mathscr{B} \to G$ over a locally compact Hausdorff groupoid G. Of course we will use the properties of $\mathscr{B}_I$ and $\mathscr{B}^I$ from Propositions 14 and 15.

We clearly have an injective $*$-homomorphism

$$\iota : \Gamma_c(G; \mathscr{B}_I) \to \Gamma_c(G; \mathscr{B})$$

given by inclusion. Also $q \mapsto q(f)$ is a $*$-homomorphism

$$q : \Gamma_c(G; \mathscr{B}) \to \Gamma_c(G; \mathscr{B}^I).$$

Proposition 15 implies that $q(\Gamma_c(G; \mathscr{B}))$ is dense in $\Gamma_c(G; \mathscr{B}^I)$ in the inductive limit topology. Therefore q has dense range when viewed a map into $C^*(G, \mathscr{B}^I)$.

Lemma 16. *The map ι extends to an isomorphism of $C^*(G, \mathscr{B}_I)$ onto an ideal* $\mathrm{Ex}(I) := \overline{\iota(\Gamma_c(G; \mathscr{B}_I))}$ *in $\Gamma_c(G; \mathscr{B})$.*

Proof. Suppose that $f \in \Gamma_c(G; \mathscr{B}_I)$ and that $g \in \Gamma_c(G; \mathscr{B})$. Then, using Lemma 10, $f(x)g(x^{-1}y) \in B(x)I(s(x))B(x^{-1}y) = B(y)I(s(y)) = B_I(y)$, and $f^*(x) \in I(s(x))B(x^{-1}) = B(x^{-1})I(r(x)) = B_I(x^{-1})$. It now follows easily that $\mathrm{Ex}(I)$ is an ideal in $C^*(G, \mathscr{B})$. We just need to see that ι is isometric for the universal norms. Let L be an *irreducible* representation of $C^*(G, \mathscr{B})$. Then either $L(\iota(\Gamma_c(G; \mathscr{B}_I))) = \{0\}$ or L defines a representation, L' of $\Gamma_c(G; \mathscr{B}_I)$ in the sense of [23, Definition 4.7] (which is nondegenerate because L is irreducible). Then

$$\|L(\iota(f))\| = \|L'(f)\| \leq \|f\|.$$

Since $\|L(\iota(f))\| \leq \|f\|$ holds for all irreducible representations, we have

$$\|\iota(f)\| \leq \|f\|.$$

Now let L' be a faithful representation of $C^*(G, \mathscr{B}_I)$ on $\mathcal{H}$. Let $\mathcal{H}_0$ be the dense subspace

$$\mathcal{H}_0 = \mathrm{span}\{L'(f)h : f \in \Gamma_c(G; \mathscr{B}_I) \text{ and } h \in \mathcal{H}\}.$$

Suppose that $f_1, \dots, f_k \in \Gamma_c(G; \mathscr{B}_I)$ and that $h_1, \dots, h_k \in \mathcal{H}$ are such that

(6) $$\sum_{i=1}^k L'(f)h_i = 0.$$

Let $g \in \Gamma_c(G; \mathscr{B})$. Let $\{e_j\}$ be an approximate identity for $\Gamma_c(G; \mathscr{B}_I)$ in the inductive limit topology (see [23, Proposition 5.1]). Then, since convolution is

continuous in the inductive limit topology, $g * e_j * f_i \to g * f_i$ in the inductive limit topology. Thus, using (6),

$$\sum_{i=1}^{k} L'(g * f_i)h_i = \lim_j \sum_{i=1}^{k} L'(g * e_j * f_i)h_i = \lim_j L'(g * e_j) \sum_{i=1}^{k} L'(f_i)h_i = 0.$$

Thus we get a well-defined homomorphism L from $\Gamma_c(G; \mathscr{B})$ to the linear operators $\mathrm{Lin}(\mathcal{H}_0)$ on $\mathcal{H}_0$ characterized by

$$L(g)L'(f)h := L'(g * f)h.$$

(Notice that $L(\iota(f))$ is just the restriction of $L'(f)$ to $\mathcal{H}_0$.) We claim that L is what we called a *pre-representation* of $\mathscr{B}$ in [23, Definition 4.1]. To see this, notice that if $g_i \to g$ in the inductive limit topology on $\Gamma_c(G; \mathscr{B})$, then $g_i * f \to g * f$ in the inductive limit topology on $\Gamma_c(G; \mathscr{B}_I)$ for any $f \in \Gamma_c(G; \mathscr{B}_I)$. Therefore

$$g \mapsto (L(g)v \mid w)$$

is continuous in the inductive limit topology for all $v, w \in \mathcal{H}_0$. Therefore condition (a) of [23, Definition 4.1] is satisfied. To verify condition (b), just note that

$$\big(L(g)L'(f)h \mid L'(f')h'\big) = \big(L'(g * f)h \mid L'(f')h'\big)$$

which, since L' is a $*$-homomorphism and since $(g * f)^* = f^* * g^*$, is

$$= \big(h \mid L'(f^* * g^* * f')h'\big)$$
$$= \big(L'(f)h \mid L(g^*)L'(f')h'\big).$$

Lastly, condition (c) follows easily from the existence of an approximate identity for $\Gamma_c(G; \mathscr{B})$ in the inductive limit topology. Now the Disintegration Theorem ([23, Theorem 4.13]) implies that L can be extended to a bounded representation of $C^*(G, \mathscr{B})$. But then

$$\|f\| = \|L'(f)\| = \|L(\iota(f))\| \le \|\iota(f)\|.$$

Thus ι is isometric as claimed. $\qquad\square$

Lemma 17. *The $*$-homomorphism q is bounded and extends to a surjective homomorphism q of $C^*(G, \mathscr{B})$ onto $C^*(G, \mathscr{B}^I)$.*

Proof. Let L' be a faithful representation of $C^*(G, \mathscr{B}^I)$. By the Disintegration Theorem [23, Theorem 4.13], we can assume that L' is the integrated form of a strict representation $(\mu, G^{(0)} * \mathscr{H}, \hat{\pi}')$, where μ is a quasi-invariant measure, $G^{(0)} * \mathscr{H}$ is a Borel Hilbert bundle and $\hat{\pi}'$ is a Borel $*$-functor on $\mathscr{B}^I$. Thus $\hat{\pi}'([b]) = \big(r(b), \bar{\pi}([b]), s(b)\big)$ for a bounded operator $\bar{\pi}([b]) : \mathcal{H}(s(b)) \to \mathcal{H}(r(b))$ with $\|\bar{\pi}([b])\| \le \|[b]\|$. Then we can define $\hat{\pi}(b) = \big(r(b), \pi(b), s(b)\big)$, where $\pi(b) := \bar{\pi}([b])$. Then $(\mu, G^{(0)} * \mathscr{H}, \hat{\pi})$ is a strict representation of $\mathscr{B}$, and its integrated form, L, satisfies $L = L' \circ q$. In particular,

$$\|q(f)\| = \|L'(q(f))\| = \|L(f)\| \le \|f\|.$$

Hence q is norm decreasing on $\Gamma_c(G; \mathscr{B})$. $\qquad\square$

Theorem 18. *Suppose that $p : \mathscr{B} \to G$ is a Fell bundle over a locally compact Hausdorff groupoid G. Let $A = \Gamma_0(G^{(0)}; \mathscr{B})$ be the C^*-algebra over $G^{(0)}$ and suppose*

that I is a G-invariant ideal in A. Let $\mathscr{B}_I$ and $\mathscr{B}^I$ be the Fell bundles described above. Then

$$0 \longrightarrow C^*(G, \mathscr{B}_I) \overset{\iota}{\longrightarrow} C^*(G, \mathscr{B}) \overset{q}{\longrightarrow} C^*(G, \mathscr{B}^I) \longrightarrow 0$$

is a short exact sequence of C^*-algebras.

In view of Lemmas 16 and 17, it will suffice to see that $\ker q = \mathrm{Ex}(I)$. This will require some work, and we start with some preliminary comments.

Let $a \in \tilde{A}$ and $f \in \Gamma_c(G; \mathscr{B})$. Define $i_A(a)f \in \Gamma_c(G; \mathscr{B})$ by

$$(i_A(a)f)(x) = a(r(x))f(x).$$

Now view $\Gamma_c(G; \mathscr{B})$ as a dense subspace of the $C^*(G, \mathscr{B})$ viewed as a right Hilbert module over itself with respect to the inner product

$$\langle f , g \rangle_* := f^* * g \quad \text{for } f, g \in \Gamma_c(G; \mathscr{B}).$$

Then it is easy to check that

$$\langle i_A(a)f , g \rangle_* = \langle f , i_A(a^*)g \rangle_* .$$

Then if $\|a\|^2 1_A - a^* a = c^* c$, we have

$$\|a\|^2 \langle f , f \rangle_* - \langle i_A(a)f , i_A(a)f \rangle_* = \langle i_A(c)f , i_A(c)f \rangle_* \geq 0.$$

Therefore i_A is bounded and extends to a homomorphism $i_A : A \to M(C^*(G, \mathscr{B}))$.

Let L be a representation of $C^*(G, \mathscr{B})$ on $\mathcal{H}$. In view of [23, Theorem 4.13], we can assume that L is the integrated form of a strict representation $(\mu, G^{(0)} * \mathscr{H}, \hat{\pi})$, where μ is a quasi-invariant measure, $\mathcal{H} = G^{(0)} * \mathscr{H}$ is a Borel Hilbert bundle and $\hat{\pi}$ is a Borel $*$-functor with $\hat{\pi}(b) = \big(r(b), \pi(b), s(b)\big)$ for an operator $\pi(b) : B(\mathcal{H}(s(b))) \to B(\mathcal{H}(r(b)))$ with $\|\pi(b)\| \leq \|b\|$. (We will often write $\pi(b)$ for both the operator and the corresponding element of $\mathrm{End}(G^{(0)} * \mathscr{H})$ defined in [23, Definition 4.3].)

Then L defines a representation π_L of A on $\mathcal{H}$ via composition with $i_A : A \to M(C^*(G, \mathscr{B}))$. It is not hard to check that if $h, k \in L^2(G^{(0)} * \mathscr{H}, \mu)$, then

$$\big(\pi_L(a)h \mid k\big) = \int_G \big(\pi(a(u))h(u) \mid k(u)\big) \, d\mu(u).$$

In particular,

$$\pi_L = \int_G^{\oplus} \pi_u \, d\mu(u),$$

where π_u is the representation of A given by $a \mapsto \pi(a(u))$.

Proof of Theorem 18. Clearly,

(7) $$\mathrm{Ex}(I) \subset \ker q.$$

Therefore it will suffice see that given any representation L of $C^*(G, \mathscr{B})$, we have either $\ker q \subset \ker L$, or $\mathrm{Ex}(I) \not\subset \ker L$.

So let L be a representation of $C^*(G, \mathscr{B})$. Let $(\mu, G^{(0)} * \mathscr{H}, \hat{\pi})$ and π_L be as above. There are two cases to consider. First, $I \subset \ker \pi_L$, and second $I \not\subset \ker \pi_L$.

Case $I \subset \ker \pi_L$. Since A, and hence I, are separable, there is a Borel null set $N \subset G^{(0)}$ such that

$$I \subset \ker \pi_u \quad \text{for all } u \notin N.$$

Let $F := G^{(0)} \setminus N$. Then if $u \in F$, we have $\pi(I(u)) = \{ 0 \}$.

Suppose that $s(x) \in F$. Since $B_I(x) = B(x) \cdot I(s(x))$ is an $I(r(x)) - I(s(x))$-imprimitivity bimodule, given any $b \in B_I(x)$, we have $b^*b \in I(s(x))$ and $0 = \pi(b^*b) = \pi(b)^*\pi(b)$. Therefore $\pi(b) = 0$, and $\pi(bb^*) = 0$. Since elements of the form bb^* space a dense subspace of $I(r(x))$, we see that $\pi(I(r(x))) = \{ 0 \}$. Furthermore, since G is σ-compact, we can shrink F a bit if necessary, and assume its saturation is Borel (see that last paragraph of the proof of [23, Lemma 5.20]). Therefore, we may as well assume that F itself is saturated.

Since μ is quasi-invariant, the restriction $G|_F$ is ν-conull.[9] Furthermore, $x \in G|_F$ and $b \in B_I(x)$ implies that $\pi(b) = 0$. Thus if $b, b' \in B(x)$ are such that $b - b' \in B_I(x)$, then $\pi(b) = \pi(b')$. Thus if $[b]$ is the class of $b \in B(x)$ in $B^I(x)$, then we can define $\bar{\pi}(x) \in B\big(\mathcal{H}(s(x)), \mathcal{H}(r(x))\big)$ by

$$\bar{\pi}([b]) = \pi(b).$$

Now we define $\hat{\pi}' : \mathscr{B}^I \to \operatorname{End}(G^{(0)} * \mathscr{H})$ by $\hat{\pi}'([b]) = \big(r(b), \bar{\pi}([b]), s(b)\big)$ where

$$\bar{\pi}([b]) = \begin{cases} \pi(b) & \text{if } p(b) \in G|_F, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It is immediate that if $f \in \Gamma_c(G; \mathscr{B})$, then $x \mapsto \hat{\pi}'(q(f)(x))$ is Borel. Since any $\bar{f} \in \Gamma_c(G; \mathscr{B}^I)$ is the uniform limit of sections of the form $q(f)$ with $f \in \Gamma_c(G; \mathscr{B})$, it follows that $x \mapsto \hat{\pi}'(\bar{f}(x))$ is Borel for all $\bar{f} \in \Gamma_c(G; \mathscr{B}^I)$. Note that since F, and hence $N = G^{(0)} \setminus F$, are saturated, G is the disjoint union of the restrictions $G|_F$ and $G|_N$. Therefore, $\bar{\pi}([a][b]) = \bar{\pi}([a])\bar{\pi}([b])$ if $([a], [b]) \in (\mathscr{B}^I)^{(2)}$. Similarly, axioms (a) and (c) of [23, Definition 4.5] are clearly satisfied and $\hat{\pi}'$ is a Borel $*$-functor on $\mathscr{B}^I$. Furthermore, $\big(\mu, G^{(0)} * \mathscr{H}, \hat{\pi}'\big)$ is a strict representation of $\mathscr{B}^I$, and since $G|_F$ is ν-conull, it follows from the Equation (4.4) in [23] that the integrated form L' satisfies $L = L' \circ q$. In particular, $\ker q \subset \ker L$ in this case.

Case $I \not\subset \ker \pi_L$. In this case, there is a $a \in I$ such that $\pi_L(a) \neq 0$. Since L is nondegenerate, there is a $f \in \Gamma_c(G; \mathscr{B})$ such that $\pi_L(a)L(f) \neq 0$. But then $L(i_A(a)f) \neq 0$. However,

$$(i_A(a)f)(x) = a(r(x))f(x) \in I(r(x))B(x) = B(x)I(s(x)) = B_I(x).$$

Therefore $i_A(a)f \in \operatorname{Ex}(I)$, and

$$\operatorname{Ex}(I) \not\subset \ker L$$

in this case. This completes the proof. $\qquad \square$

References

[1] Beatriz Abadie and Ruy Exel, *Deformation quantization via Fell bundles*, Math. Scand. **89** (2001), 135–160.

[2] A. Buss, R. Meyer, and C. Zhu, *A higher category approach to twisted actions on C*-algebras*, preprint, 2009. (arXiv:math.OA.0908.0455v1).

[3] Alcides Buss, *Integrability of dual coactions on Fell bundle C*-algebras*, preprint, 2009. (arXiv:math.OA.0908.0559).

---

[9] Recall that ν is the measure on G given by $\int_{G^{(0)}} \lambda^u \, d\mu(u)$.

[4] Alcides Buss and Ruy Exel, *Fell bundles over inverse semigroups and twisted étale groupoids*, preprint, 2009. (arXiv:math.OA.0903.3388v3).

[5] Valentin Deaconu, Alex Kumjian, and Birant Ramazan, *Fell bundles and groupoid morphisms*, Math. Scan. **103** (2008), 305–319.

[6] Maurice J. Dupré and Richard M. Gillette, *Banach bundles, Banach modules and automorphisms of C^*-algebras*, Vol. 92, Pitman (Advanced Publishing Program), Boston, MA, 1983.

[7] Ruy Exel, *Amenability for Fell bundles*, J. Reine Angew. Math. **492** (1997), 41–73.

[8] ———, *Twisted partial actions: a classification of regular C^*-algebraic bundles*, Proc. London Math. Soc. (3) **74** (1997), 417–443.

[9] ———, *Partial representations and amenable Fell bundles over free groups*, Pacific J. Math. **192** (2000), 39–63.

[10] ———, *Exact groups and Fell bundles*, Math. Ann. **323** (2002), 259–266.

[11] Ruy Exel and Marcelo Laca, *Continuous Fell bundles associated to measurable twisted actions*, Proc. Amer. Math. Soc. **125** (1997), 795–799.

[12] James M. G. Fell and Robert S. Doran, *Representations of ∗-algebras, locally compact groups, and Banach ∗-algebraic bundles. Vol. 1*, Pure and Applied Mathematics, vol. 125, Academic Press Inc., Boston, MA, 1988. Basic representation theory of groups and algebras.

[13] ———, *Representations of ∗-algebras, locally compact groups, and Banach ∗-algebraic bundles. Vol. 2*, Pure and Applied Mathematics, vol. 126, Academic Press Inc., Boston, MA, 1988. Banach ∗-algebraic bundles, induced representations, and the generalized Mackey analysis.

[14] Igor Fulman, *Coordinatization for Fell bundle algebras*, Houston J. Math. **24** (1998), 313–324.

[15] Igor Fulman and Paul S. Muhly, *Bimodules, spectra, and Fell bundles*, Israel J. Math. **108** (1998), 193–215.

[16] Philip Green, *The local structure of twisted covariance algebras*, Acta Math. **140** (1978), 191–250.

[17] Karl Heinrich Hofmann, *Banach bundles*, 1974. Darmstadt Notes.

[18] ———, *Bundles and sheaves are equivalent in the category of Banach spaces*, K-theory and operator algebras (Proc. Conf., Univ. Georgia, Athens, Ga., 1975), Lecture Notes in Math, vol. 575, Springer, Berlin, 1977, pp. 53–69.

[19] Marius Ionescu and Dana P. Williams, *A classic Morita equivalence result for Fell bundle C^*-algebras*, Math. Scan. (2010), in press. (arXiv:math.OA.0912.1125).

[20] Steven Kaliszewski, Paul S. Muhly, John Quigg, and Dana P. Williams, *Coactions and Fell bundles*, preprint, 2009. (arXiv:math.OA.0909.3259).

[21] Alex Kumjian, *Fell bundles over groupoids*, Proc. Amer. Math. Soc. **126** (1998), 1115–1125.

[22] E. C. Lance, *Hilbert C^*-modules*, London Mathematical Society Lecture Note Series, vol. 210, Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.

[23] Paul S. Muhly and Dana P. Williams, *Equivalence and disintegration theorems for Fell bundles and their C^*-algebras*, Dissertationes Math. (Rozprawy Mat.) **456** (2008), 1–57.

[24] Iain Raeburn and Dana P. Williams, *Pull-backs of C^*-algebras and crossed products by certain diagonal actions*, Trans. Amer. Math. Soc. **287** (1985), 755–777.

[25] ———, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, vol. 60, American Mathematical Society, Providence, RI, 1998.

[26] Dana P. Williams, *Crossed products of C^*-algebras*, Mathematical Surveys and Monographs, vol. 134, American Mathematical Society, Providence, RI, 2007.

[27] Shigeru Yamagami, *On the ideal structure of C^*-algebras over locally compact groupoids*, 1987. (Unpublished manuscript).

[28] ———, *On primitive ideal spaces of C^*-algebras over certain locally compact groupoids*, Mappings of operator algebras (Philadelphia, PA, 1988), Progr. Math., vol. 84, Birkhäuser Boston, Boston, MA, 1990, pp. 199–204.

Department of Mathematics, University of Connecticut, Storrs, CT 06269-3009
E-mail address: `ionescu@math.uconn.edu`

Department of Mathematics, Dartmouth College, Hanover, NH 03755-3551
E-mail address: `dana.williams@Dartmouth.edu`