3.8 Exponential Growth and Decay

Marius Ionescu

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Population growth

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- In nuclear physics, the mass of a radioactive substance decays at a rate proportional to the mass.

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- It is an example of a ***differential equation***.

Solutions

Fact

The only solution of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}.$$

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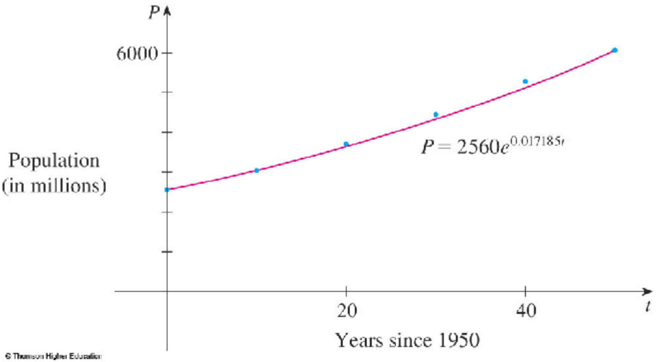
  where P is the population?
- Use the model to estimate the population in 1993 and to predict the population in 2020.

Population growth



Radioactive decay

- If $m(t)$ us the mass remaining from an initial mass m_0 if the substance after time t, then the **relative decay rate**

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- The **half-life** is the time required for half of any given quantity to decay.

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- A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.
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- When will the mass be reduced to 20 mg?

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- **Newton's Law of Cooling** is the following differential equations

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- Set $y(t) = T(t) - T_s$; then the equation becomes

$$\frac{dy}{dt} = ky.$$

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- What is the temperature of the soda pop after another half hour?
- How long does it take for the soda pop to cool to 50°F?

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A freshely brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

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1. Find the mass that remains after t years.
2. How much of the sample remains after 100 years?
3. After how long will only 1 mg remain?