3.8 Exponential Growth and Decay

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October 15, 2010

Population growth

Example

- If $y = f(t)$ is the number of individuals in a population of animals or humans at time t, then it seems reasonable to expect that the rate of growth $f'(t)$ is proportional to the population.
- In nuclear physics, the mass of a radioactive substance decays at a rate proportional to the mass.

Differential equations

- Then the rate of change of y with respect to t satisfies the equation
$$\frac{dy}{dt} = ky,$$
where k is a constant.
- If $k > 0$ the equation is called the **law of natural growth**.
- If $k < 0$ the equation is called the **law of natural decay**.
- It is an example of a ***differential equation***.

Solutions

Fact

The only solution of the differential equation $dy/dt = ky$ are the exponential functions

$$y(t) = y(0)e^{kt}.$$

Population growth

Example

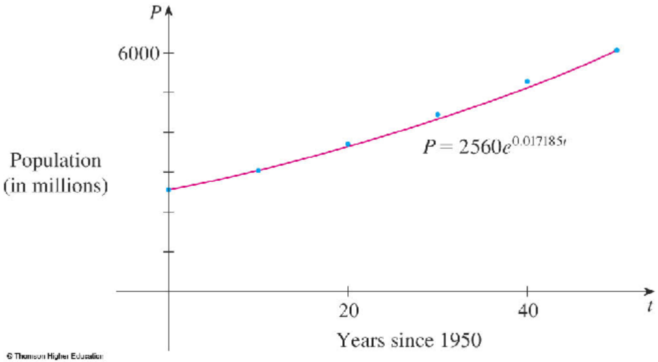
Use the fact that the world population was 2,560 million in 1950 and 3,040 million in 1960 to model the population in the second half of the 20th century.

- What is the relative growth rate

$$\frac{1}{P}\frac{dP}{dt},$$

  where P is the population?
- Use the model to estimate the population in 1993 and to predict the population in 2020.

Population growth



Radioactive decay

- If $m(t)$ us the mass remaining from an initial mass m_0 if the substance after time t, then the **relative decay rate**

$$-\frac{1}{m}\frac{dm}{dt} = -k,$$

where k is a negative constant.

- So $m(t)$ decays exponentially

$$m(t) = m_0 e^{kt}.$$

- The **half-life** is the time required for half of any given quantity to decay.

Example

Example

The half-life of radium-226 is 1590 years.

- A sample of radium-226 has a mass of 100 mg. Find a formula for the mass of the sample that remains after t years.
- Find the mass after $1,000$ years correct to the nearest milligram.
- When will the mass be reduced to 20 mg?

Newton's Law of Cooling

- Let $T(t)$ be the temperature of the object at time t and T_s be the temperature of the surroundings.
- **Newton's Law of Cooling** is the following differential equations
$$\frac{dT}{dt} = k(T - T_s).$$
- Set $y(t) = T(t) - T_s$; then the equation becomes
$$\frac{dy}{dt} = ky.$$

Example

A bottle of soda pop at room temperature (72°F) is placed in a refrigerator, where the temperature is 44°F. After half an hour, the soda pop has cooled to 61°F.

- What is the temperature of the soda pop after another half hour?
- How long does it take for the soda pop to cool to 50°F?

Example

Example

A freshely brewed cup of coffee has temperature 95°C in a 20°C room. When its temperature is 70°C, it is cooling at a rate of 1°C per minute. When does this occur?

Example

Example

The half-life of cesium-137 is 30 years. Suppose we have a 100-mg sample.

1. Find the mass that remains after t years.
2. How much of the sample remains after 100 years?
3. After how long will only 1 mg remain?