4.3 How Derivatives Affect the Shape of a Graph

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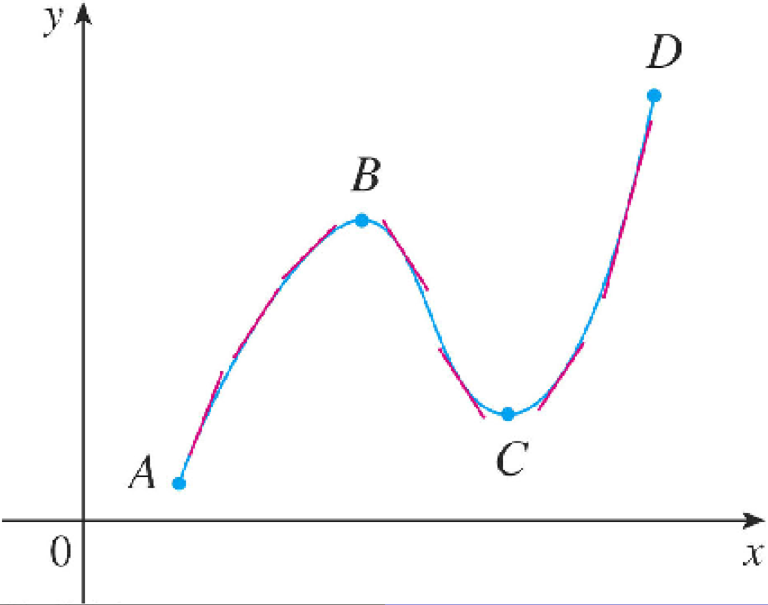
What does f' say about f?



Increasing/Decreasing Test

Fact

- *If* $f'(x) > 0$ *on an interval, then* f *is increasing on that interval.*
- *If* $f'(x) < 0$ *on an interval, then* f *is decreasing on that interval.*

Example

Find the intervals on which $f(x)$ is increasing or decreasing

- $f(x) = 4x^3 + 3x^2 - 6x + 1$
- $f(x) = x^2 \ln x$.

The First Derivative Test

Fact

Suppose that c is a critical number of a continuous function f.

1. *If f' changes from positive to negative at c, then f has a local maximum at c.*
2. *If f' changes from negative to positive at c, then f has a local minimum at c.*
3. *If f' does not change sign at c—for example, if f' is positive on both sides of c or negative on both sides—then f has no local maximum or minimum at c.*

Example

Find the local maximum and minimum values of f:

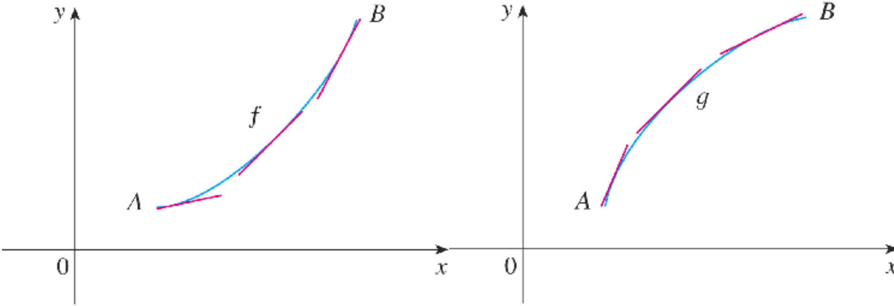
- $f(x) = 4x^3 + 3x^2 - 6x + 1$
- $f(x) = x^2 \ln x$.

What Does f'' say about f ?

Definition

- If the graph of f lies above all of its tangents on an interval I, it is called concave *upward* on I.
- If the graph of f lies below all of its tangents on I, it is called concave *downward* on I.

Example



(a) Concave upward

(b) Concave downward

Concavity Test

Fact

- *If* $f''(x) > 0$ *for all* x *in* I, *then the graph of* f *is concave upward on* I.
- *If* $f''(x) < 0$ *for all* x *in* I, *then the graph of* f *is concave downward on* I.

Inflection Point

Definition

A point P on a curve $y = f(x)$ is called an ***inflection point*** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

Example

Sketch a possible graph of a function f that satisfies the following conditions:

1. $f'(x) > 0$ on $(-\infty, 1)$, $f'(x) < 0$ on $(1, \infty)$.
2. $f''(x) > 0$ on $(-\infty, -2)$ and $(2, \infty)$, $f''(x) < 0$ on $(-2, 2)$.
3. $\lim_{x\to-\infty} f(x) = -4$, $\lim_{x\to\infty} f(x) = 0$.

The second derivative test

Fact

Suppose f'' is continuous near c.

- *If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.*
- *If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.*

Example

Find the local maximum and minimum values of f using the First and Second Derivative Tests

- $f(x) = \frac{x}{x^2+4}$
- $f(x) = x + \sqrt{1-x}$

Example

Example

Find the vertical and horizontal asymptotes, the intervals of increase or decrease, the local maximum and minimum values, the intervals of concavity and the inflection points for

$$f(x) = \frac{x^2}{x^2 - 1}.$$

Use the information to sketch the graph of f.