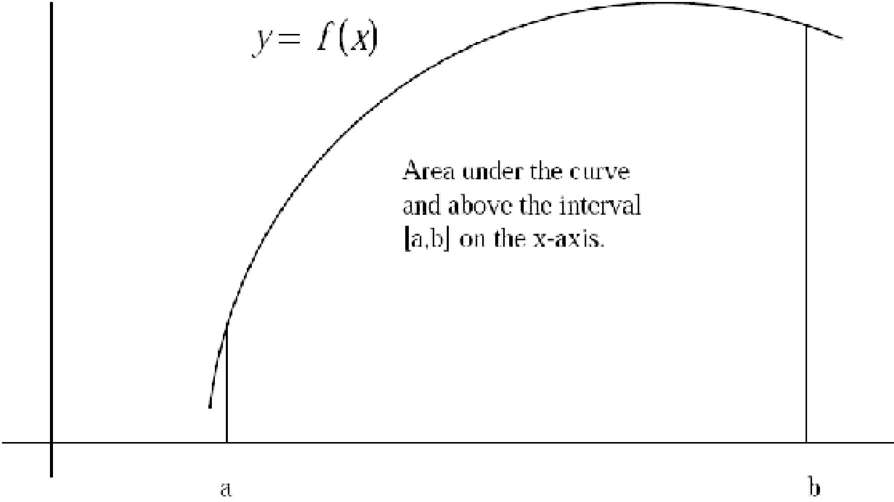
5.1 Area

11/29/2010

The Area Problem

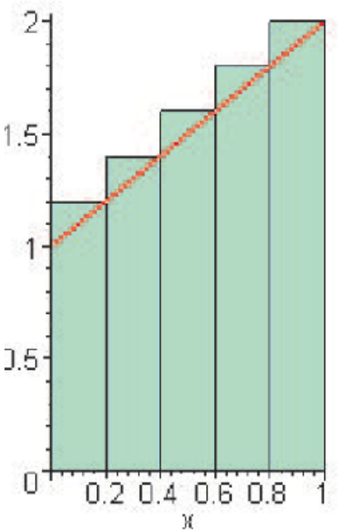


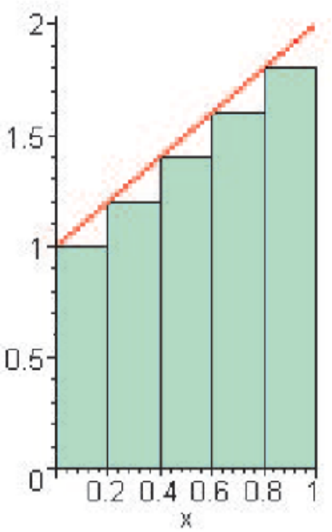
Assumptions about Areas

1. Area is a nonnegative number.
2. The area of a rectangle is its length times its width.
3. Area is additive. That is, if a region is completely divided into a finite number of non-overlapping subregions, then the area of the region is the sum of the areas of the subregions.

Upper and Lower Sums

Suppose we want to use rectangles to approximate the area under the graph of $y = x + 1$ on the interval $[0, 1]$.





Upper and Lower Sums

- We will call the sum of the areas of the rectangles in the left picture an *Upper Riemann Sum*, and the sum of the areas of the rectangles in the right picture a *Lower Riemann Sum*.
- The Upper Sum $= 31/20$ and Lower Sum $= 29/20$.

n	U	L
100	1.505000000	1.495000000
150	1.503333333	1.496666667
200	1.502500000	1.497500000
300	1.501666667	1.498333333
500	1.501000000	1.499000000

Sigma Notation

- If m and n are integers with $m \leq n$, and if f is a function defined on the integers from m to n, then the symbol

$$\sum_{i=m}^{n} f(i),$$

called sigma notation, is defined to be

$$f(m) + f(m+1) + f(m+2) + \cdots + f(n).$$

Example

1. $\sum_{i=1}^{n} i = 1 + 2 + 3 + 4 \cdots + n$
2. $\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + 3^2 + 4^2 \cdots + n^2$
3. $\sum_{i=1}^{n} 1 = \underbrace{1 + 1 + 1 + 1 \cdots + 1}_{ntimes}$

The Area Problem Revisited

$$\begin{aligned} U(P,f) &= \sum_{i=1}^{n} M_i \Delta x \\ L(P,f) &= \sum_{i=1}^{n} m_i \Delta x, \end{aligned}$$

where M_i and m_i are, respectively, the maximum and minimum values of f on the ith subinterval $[x_{i-1}, x_i]$, $1 \leq i \leq n$.

Riemann Sums

- Let f be defined on $[a, b]$.
- Divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b - a)/n$.
- Then the Right Riemann Sum is

$$R_n = \sum_{i=1}^{n} f(x_i)\Delta x,$$

and the Left Riemann Sum is

$$L_n = \sum_{i=0}^{n} f(x_i)\Delta x.$$

Area of a region

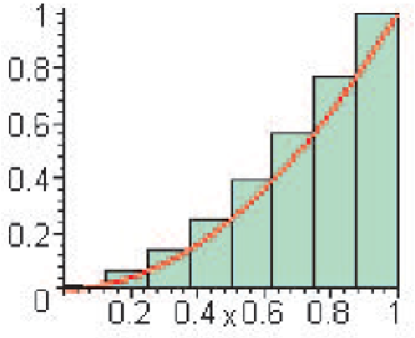
Definition

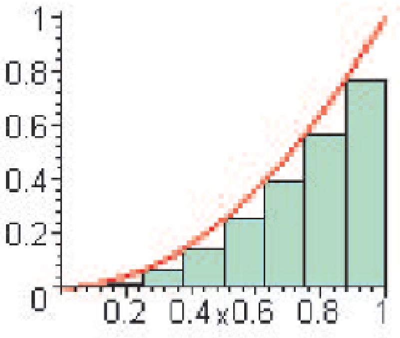
The area A of the region S that lies under the graph of the continuous function f is the limit of the sum of the areas of approximating rectangles:

$$A = \lim_{n \to \infty} R_n = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i)\Delta x.$$

Example

Show that the area under the graph of $y = f(x) = x^2$ on the interval $[0, 1]$ is $\frac{1}{3}$.





Definition

If f is a function defined for $a \leq x \leq b$, we divide the interval $[a, b]$ into n subintervals of equal width $\Delta x = (b–a)/n$.

- We let $x_0(= a), x_1, x_2, \ldots, x_n(= b)$ be the endpoints of these subintervals.
- We let $x_1^*, x_2^*, \ldots., x_n^*$ be any sample points in these subintervals, so x_i^* lies in the i th subinterval.
- Then, the definite integral of f from a to b is

$$\int_a^b f(x)dx = \lim_{nt \to \infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$

Example

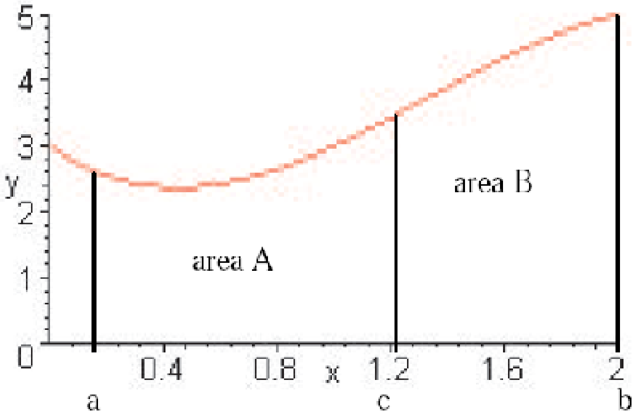
Example

Evaluate $\int_0^3 (x^3 - 6x)dx$.

Properties of the definite integral

- If f is continuous on $[a, b]$, then f is Riemann integrable on $[a, b]$.
- $\int_a^a f(x)dx = 0$.
- If f is integrable and $f(x) \geq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the area of the region under the graph of f and above the interval $[a, b]$.
- If $f(x) \leq 0$ on $[a, b]$, then $\int_a^b f(x)dx$ equals the negative of the area of the region between the interval $[a, b]$ and the graph of f.

$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx.$



Properties of the indefinite integral

- If f and g are integrable on $[a, b]$, then

$$\int_a^b (Af(x) + Bg(x))dx = A\int_a^b f(x)dx + B\int_a^b g(x)dx,$$

for any constants A and B.