2.8 The Derivative as a Function

Marius Ionescu

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Definition

- The derivative of a function f is a new function defined by

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

- We will say that a function f is differentiable at a point $x = a$ if the derivative function f' exists at a.

Example

Suppose we consider the piecewise defined function

$$f(x) = \begin{cases} x & x \leq 1 \\ 1 & 1 < x < 3 \\ -x + 4 & 3 \leq x \end{cases}$$

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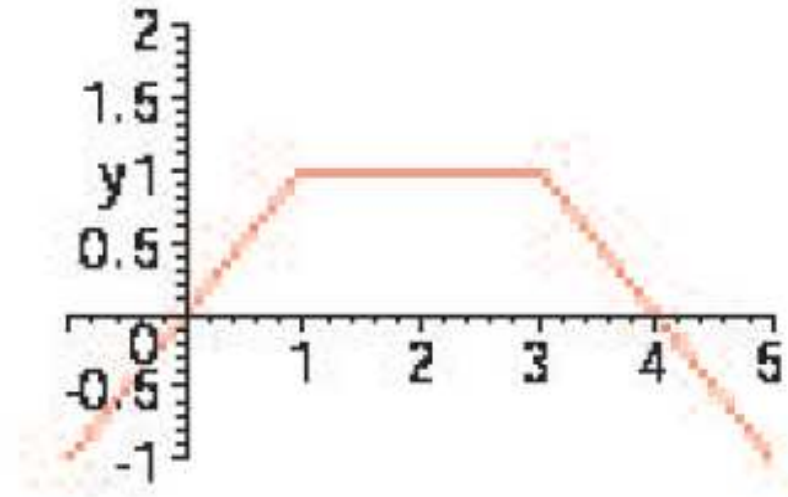
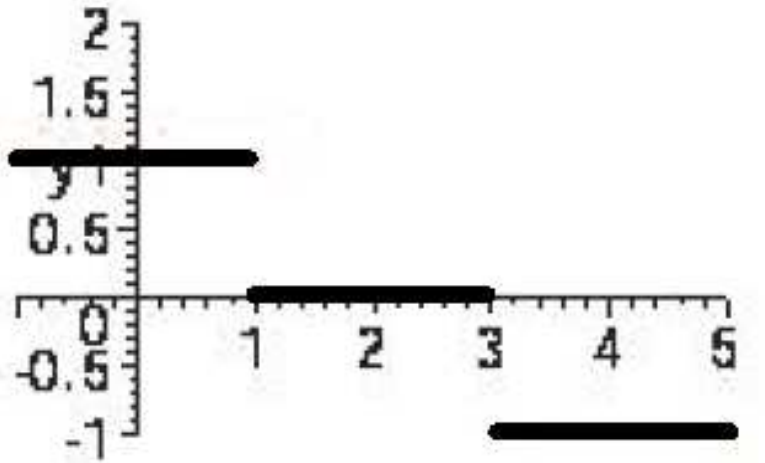
It's derivative is:

$$f(x) = \begin{cases} 1 & x < 1 \\ 0 & 1 < x < 3 \\ -1 & 3 < x \end{cases}$$

Example





Example

Example

Let

$$f(x) = k,$$

where k is a constant.

Example

Example

Let

$$f(x) = ax + b,$$

a, b constants.

The derivative of x^2

Example

For $f(x) = x^2$, we have

The derivative of x^2

Example

For $f(x) = x^2$, we have

$$f'(x) = 2x$$

The derivative of x^3

Example

For $f(x) = x^3$, we have

The derivative of x^3

Example

For $f(x) = x^3$, we have

$$f'(x) = 3x^2$$

The derivative of $1/x$

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For $f(x) = \frac{1}{x}$, we have

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Example

For $f(x) = \frac{1}{x}$, we have

$$f'(x) = -\frac{1}{x^2}$$

The derivative of $\sqrt{x}$

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The derivative of $\sqrt{x}$

Example

For $f(x) = \sqrt{x}$, we have

$$f(x) = \frac{1}{2\sqrt{x}}$$

The Power Rule

Example

Suppose that $f(x) = x^r$, where r is any real number. Then

$$f'(x) = rx^{r-1}.$$

Example

Example

Find an equation of the tangent line to the graph of $f(x) = x^{4/3}$ at the point where $x = 1$.

$$y = f(1) + f'(1)(x - 1).$$

Example

Example

Find the derivative of $f(x) = |x|$.

Example

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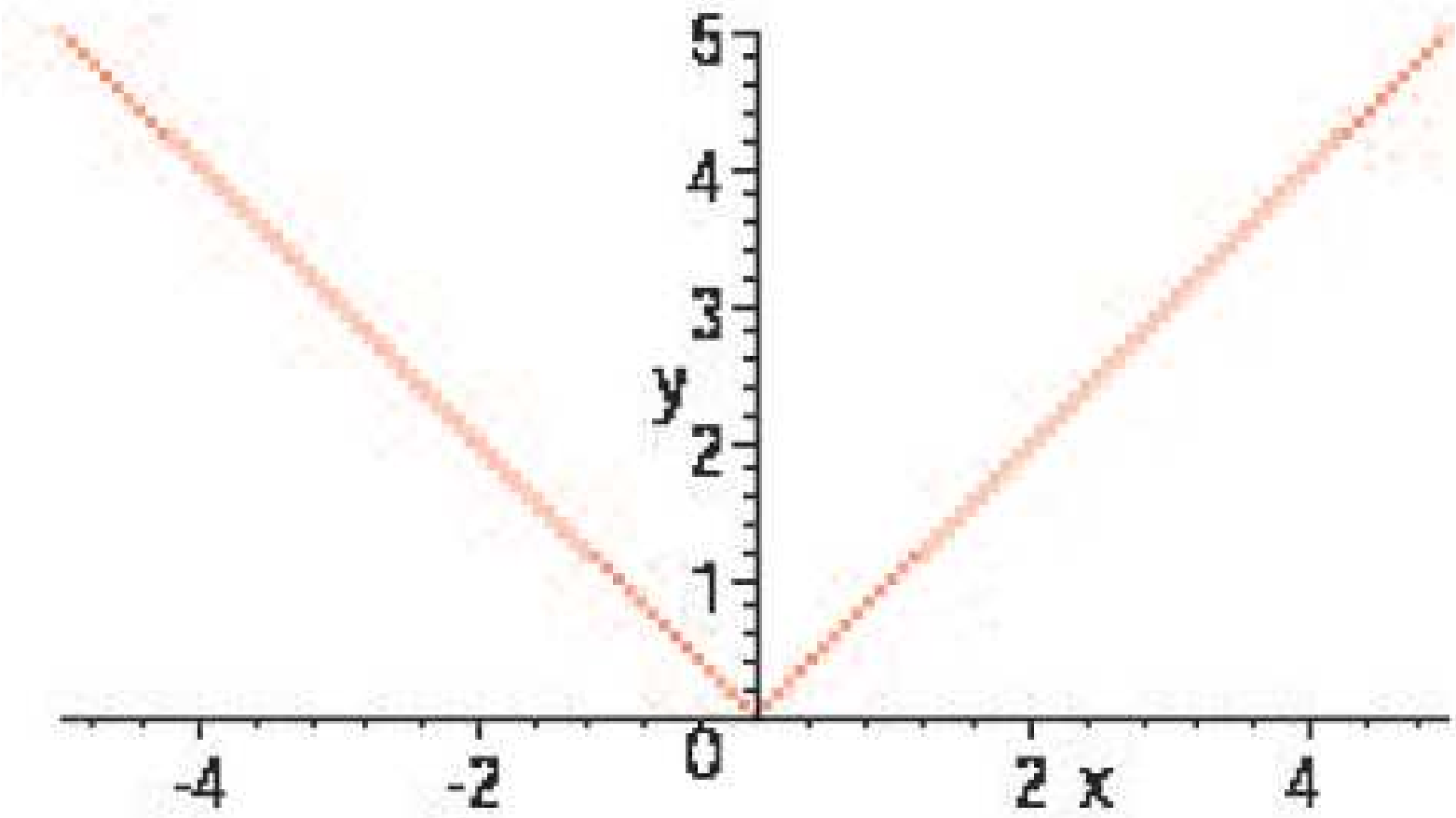
$$\lim_{h \to 0^+} \frac{|0+h| - 0}{h} = 1$$

$$\lim_{h \to 0^-} \frac{|0+h| - 0}{h} = -1$$

Absloute value



Differentiable functions and Continuous functions

Fact

Important:

- *If f is differentiable at a, then f is continuous at a.*

Differentiable functions and Continuous functions

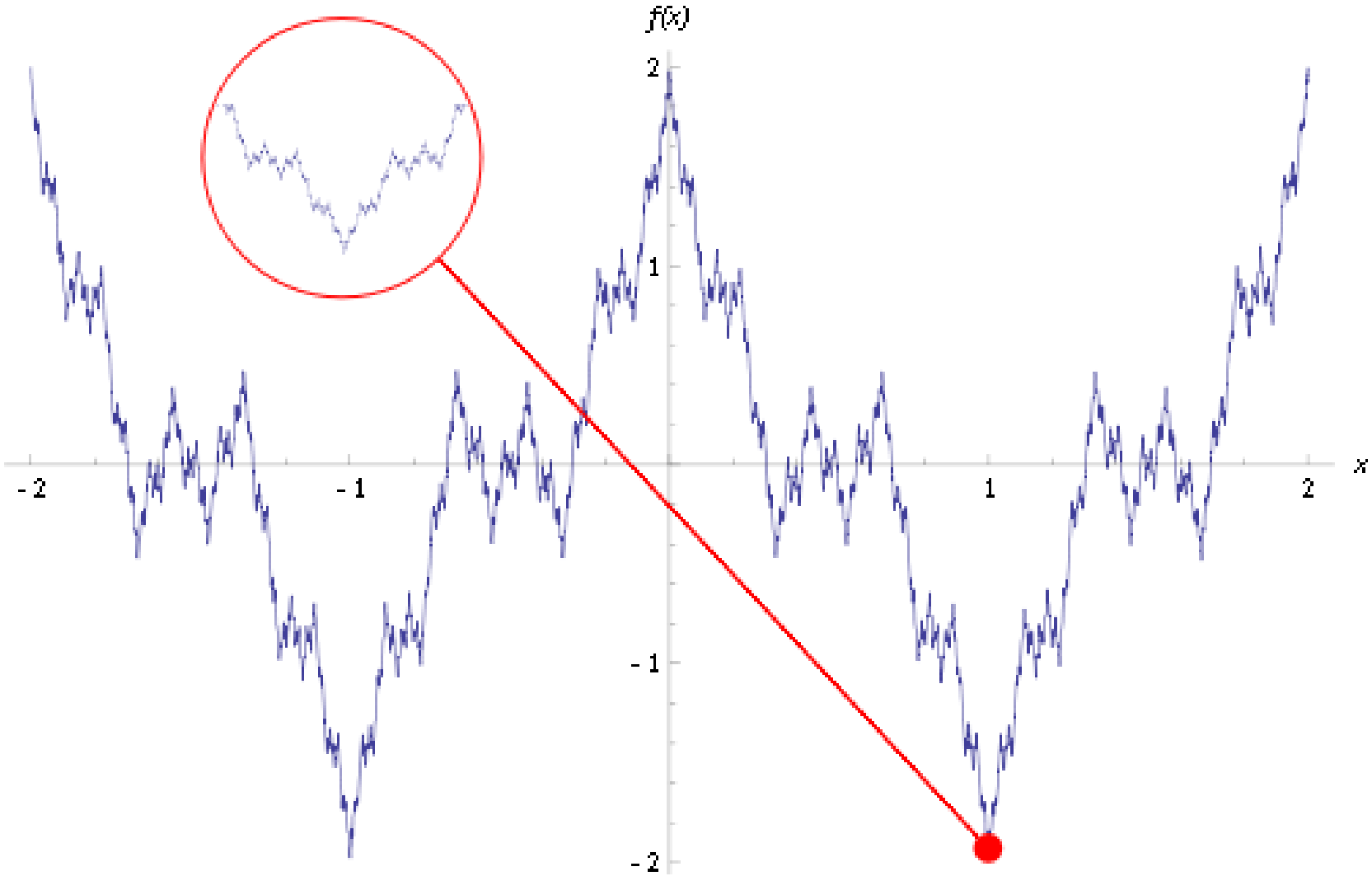
Fact

Important:

- *If f is differentiable at a, then f is continuous at a.*
- *If f is continuous at a, then f might not be differentiable at a.*

A continuous function which is nowhere differentiable (Weierstrass)

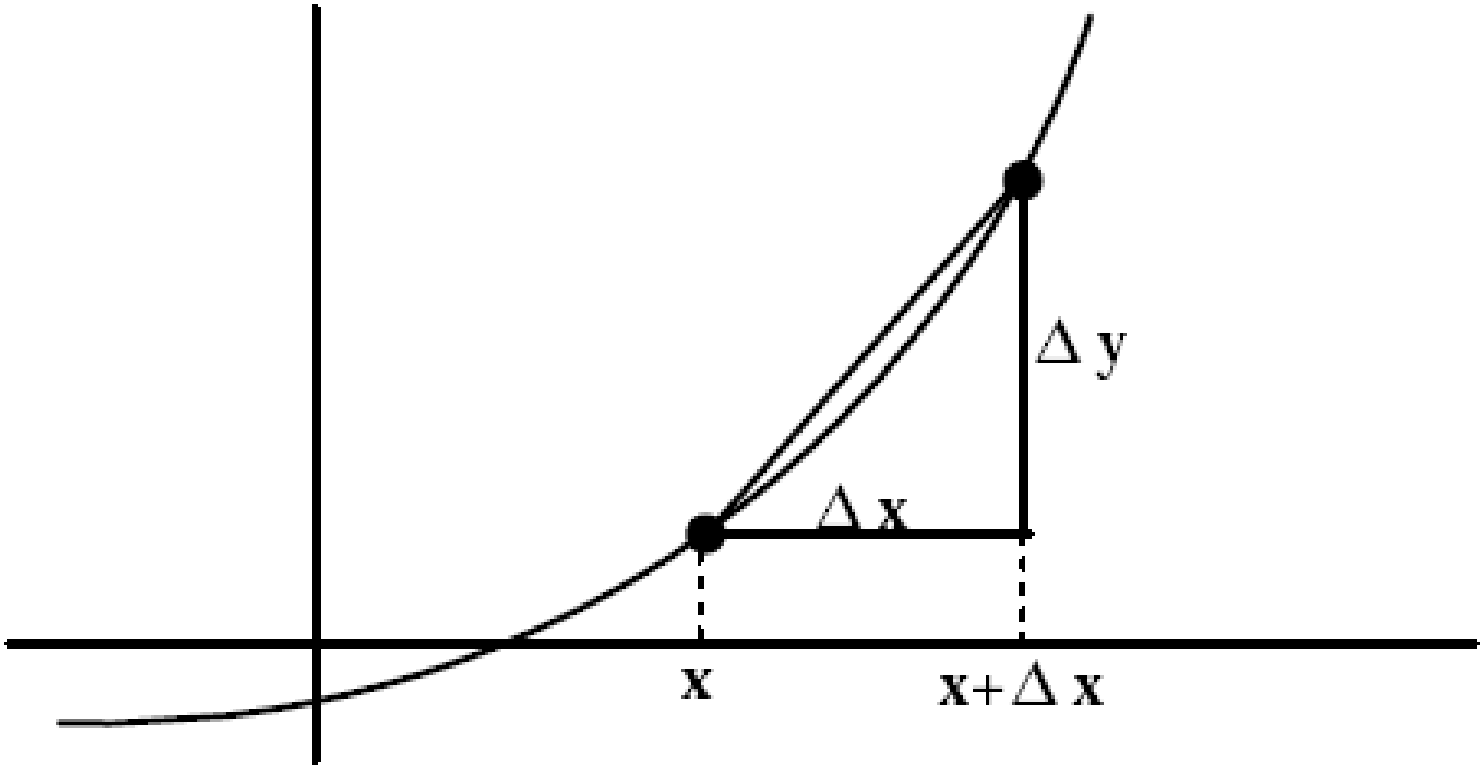
From Wikipedia



Notation for the Derivative

$$y' = D_x y = \frac{dy}{dx} = \frac{d}{dx} f(x) = f'(x).$$

The notation $\frac{dy}{dx}$



Example

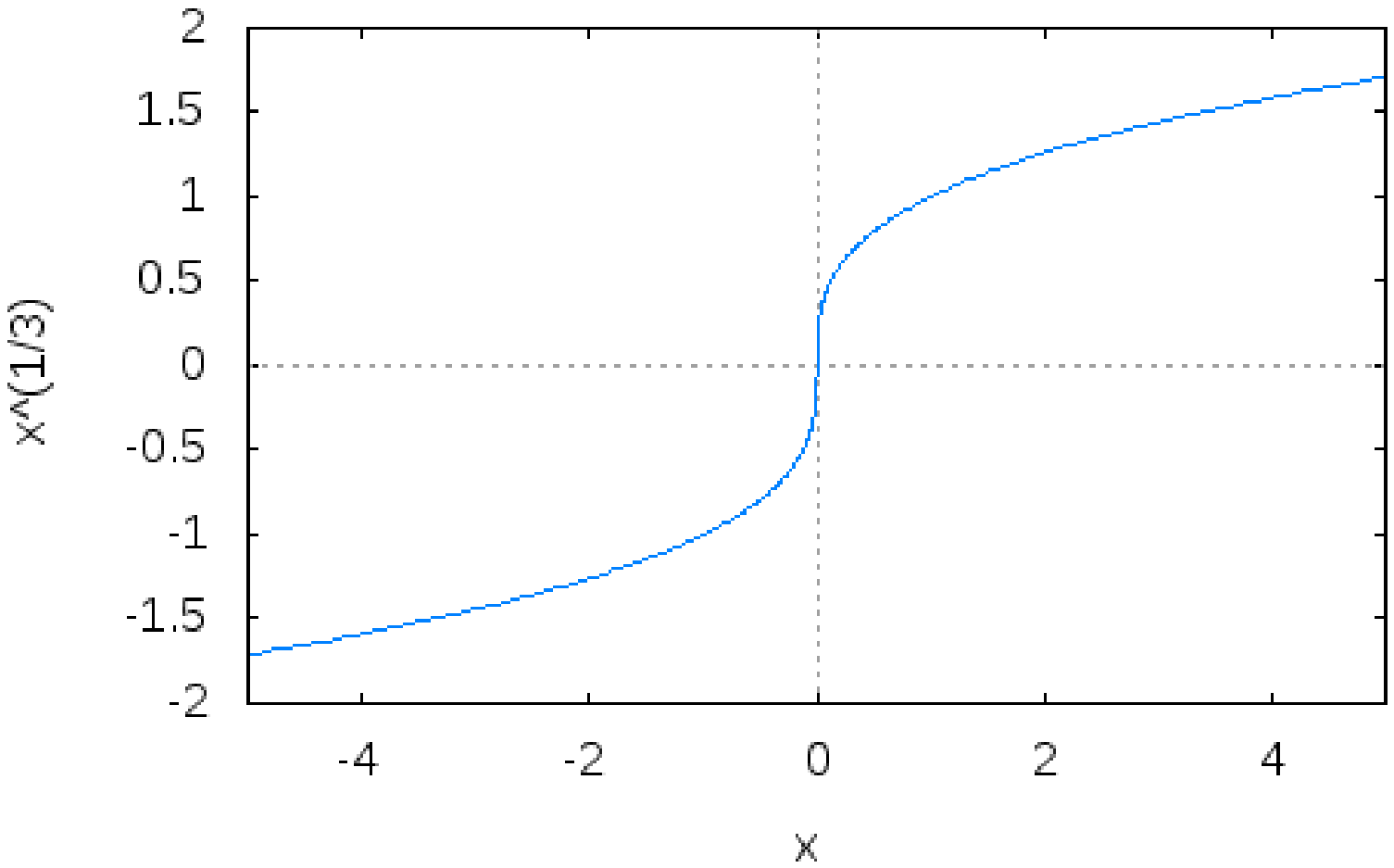
Example

For the function $y = f(x) = 1/x$, find the slope of its tangent line at $x = 2$. Compare it with the average rate of change over the interval $[2, 3]$.

Examples

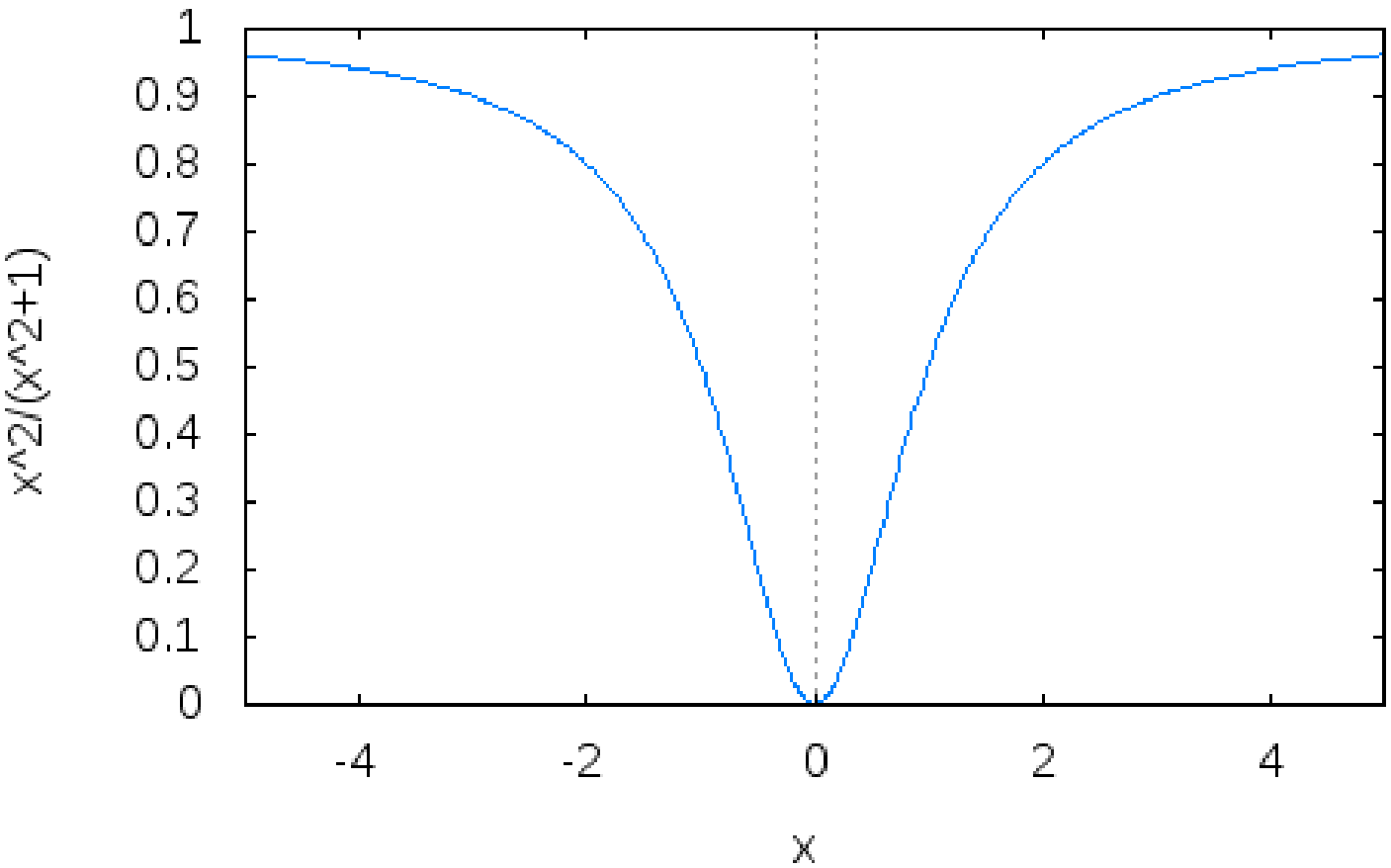
Example

Let $f(x) = \sqrt[3]{x}$. Is $f(x)$ defined at 0? Is $f(x)$ continuous at 0? Is $f(x)$ differentiable at 0?



Example

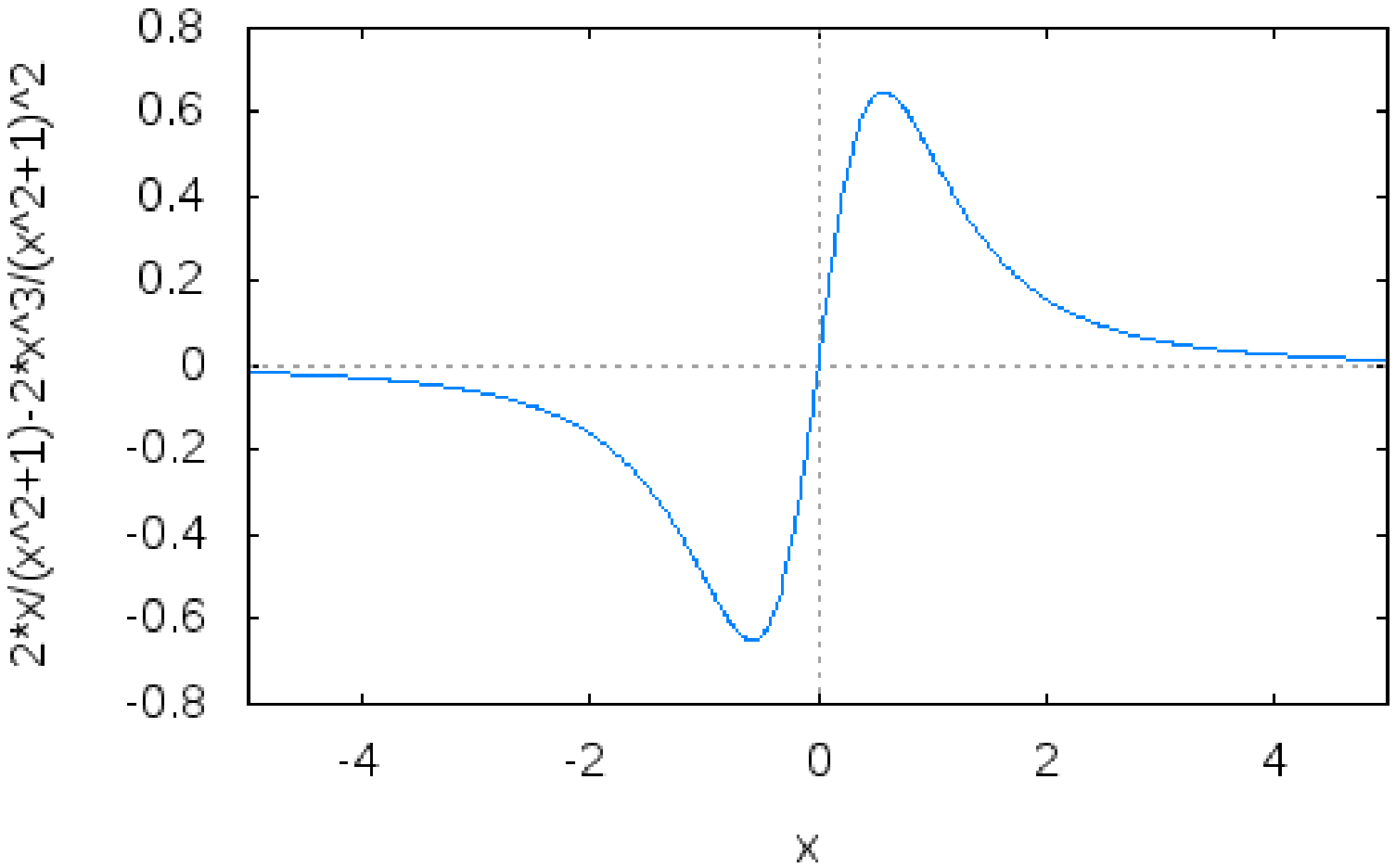
Consider the following graph of a function



Sketch the graph of f'.

Example

The graph of the derivative is



Higher Order Derivatives

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- The derivative is again a candidate for differentiation, and we call its derivative *the second derivative* of $f(x)$.
- So long as the derivatives exist we can continue this process to obtain a succession of higher derivatives.

Higher Order Derivatives

$$y'' = f''(x) = \frac{d^2y}{d^2x} = \frac{d}{dx}\frac{d}{dx}f(x) = \frac{d^2}{dx^2}f(x) = Dx^2y = Dx^2f(x).$$

The nth derivative

$$y^{(n)} = f^{(n)}(x) = \frac{d^n y}{d^n x} = \frac{d^n}{dx^n} f(x) = Dx^n y = Dx^n f(x).$$

Example

Example

Find the third derivative of the function

$$f(x) = 3x^3 + 4x^2 + 1.$$