Instructor: Marius Ionescu

1. Assume that the position of an object which is moving in the x-axis is given by $x(t) = t^2 + 3t$.

(5pts) (a) Find the average velocity between $t = 1$ and $t = 3$.

Solution: The average velocity between 1 and 3 is given by

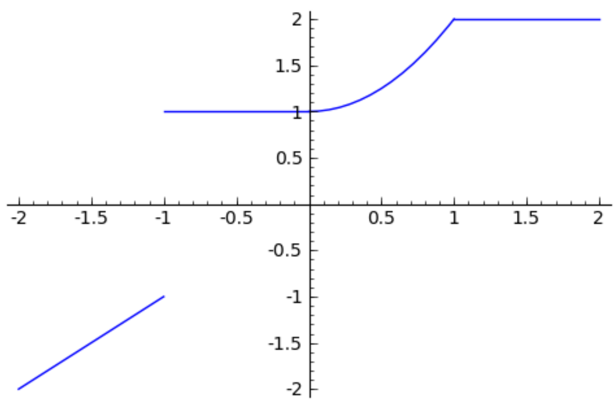
$$\frac{\Delta x}{\Delta t} = \frac{x(3) - x(1)}{3 - 1} = \frac{18 - 4}{2} = 7.$$

(5pts) (b) Find the instantaneous velocity at $t = 1$ using the definition. No credit will be given if the definition is not used.

Solution: The instantaneous velocity at $t = 1$ is the limit as h approaches 0 of the average velocity $\big(x(1+h) - x(1)\big)/h$, that is, it equals the derivative of $x(t)$ at $t = 1$. Using the definition we have

$$\begin{aligned} v(1) &= \lim_{h \to 0} \frac{x(1+h) - x(1)}{h} = \lim_{h \to 0} \frac{(1+h)^2 + 3(1+h) - 1 - 3}{h} \\ &= \lim_{h \to 0} \frac{1 + 2h + h^2 + 3 + 3h - 4}{h} = \lim_{h \to 0} \frac{h(h+5)}{h} = 5. \end{aligned}$$

(10pts) 2. Consider the following graph of a function. Describe the points of discontinuity for the function as well as the points where the function fails to be differentiable. Please explain.



Solution: The function is discontinuous at $x = -1$ since the limit from the left at -1 and the limit from the right at -1 are different (there is a jump at -1). Then the function is not differentiable at -1 either, because a differentiable function should be continuous.

Also, the function is not differentiable at $x = 1$ because we have a cusp there.

3. Compute the following limits

(5pts) (a)

$$\lim_{x\to\infty} \frac{\sqrt{3x^2+2}}{x}.$$

Solution: Using the limit laws we have

$$\begin{aligned} \lim_{x\to\infty} \frac{\sqrt{3x^2+2}}{x} &= \lim_{x\to\infty} \frac{\sqrt{x^2\left(3+\frac{2}{x^2}\right)}}{x} \\ &= \lim_{x\to\infty} \frac{x\sqrt{3+\frac{2}{x^2}}}{x} = \sqrt{3}, \end{aligned}$$

since $\lim_{x\to\infty} 2/x^2 = 0$.

(5pts) (b)

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right).$$

Explain your reasoning (no credit will be given without justification).

Solution: We discussed in class that $\sin(1/x)$ does not have a limit as x approaches 0. Therefore we can not use the limit laws (product law) to solve this problem. We are going to use the squeeze theorem. We start with the inequality

$$-1 < \sin\left(\frac{1}{x}\right) < 1$$

and we multiply all sides by x^2 to obtain

$$-x^2 < x^2 \sin\left(\frac{1}{x}\right) < x^2.$$

Since $\lim_{x\to 0} x^2 = \lim_{x\to 0} -x^2 = 0$, the squeeze theorem implies that

$$\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$$

4. Let

$$f(x) = \begin{cases} kx & x < 1 \\ x^3 + 1 & 1 < x \end{cases}.$$

(5pts) (a) Find the value of the constant k such that f has a continuous extension to $x = 1$.

Solution: Since the function is not defined for $x = 1$ we can not plug in $x = 1$ in either one of the sides. We have to use the definition of the continuity using limits. Namely, $f(x)$ has a continuous extension to $x = 1$ if

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^+} f(x).$$

Computing the two limits we have

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} kx = k,$$

and

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x^3 + 1 = 2.$$

Thus $k = 2$.

(5pts) (b) Is the continuous extension differentiable at $x = 1$? Explain your reasoning (no credit will be given without justification).

Solution: Notice that the continuous extension of $f(x)$ to 1 is given by $f(1) = 2$. To figure out if the function is differentiable at $x = 1$ we need to compute the two-sided limits for the derivative, namely:

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h}.$$

The first limit equals

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{2(1+h) - 2}{h} = 2.$$

The second limit equals

$$\begin{aligned} \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} &= \lim_{h \to 0^+} \frac{(1+h)^3 + 1 - 2}{h} \\ &= \lim_{h \to 0^+} \frac{1 + 3h + 3h^2 + h^3 - 1}{h} \\ &= \lim_{h \to 0^+} \frac{h(3 + 3h + h^2)}{h} = 3. \end{aligned}$$

Since the two limits are different, it follows that the function is not differentiable at $x = 1$.

5. Let

$$f(x) = \frac{1}{\sqrt{x+1}}.$$

(10pts) (a) Use the limit definition of the derivative function to find the derivative $f'(x)$. No credit will be given if the definition is not used.

Solution: I know that this is a long computation, but we did something similar in class. In any case, we can proceed as follows:

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{x+h+1}} - \frac{1}{\sqrt{x+1}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} = \lim_{h\to 0} \frac{\sqrt{x+1}-\sqrt{x+h+1}}{h\sqrt{x+h+1}\sqrt{x+1}} \cdot \frac{\sqrt{x+1}+\sqrt{x+h+1}}{\sqrt{x+1}+\sqrt{x+h+1}} \\
&= \lim_{h\to 0} \frac{x+1-x-h-1}{h\sqrt{x+h+1}\sqrt{x+1}(\sqrt{x+1}+\sqrt{x+h+1})} \\
&= \lim_{h\to 0} \frac{-h}{h\sqrt{x+h+1}\sqrt{x+1}(\sqrt{x+1}+\sqrt{x+h+1})} \\
&= \lim_{h\to 0} \frac{-1}{\sqrt{x+h+1}\sqrt{x+1}(\sqrt{x+1}+\sqrt{x+h+1})} \\
&= = \frac{-1}{2(x+1)\sqrt{x+1}}.
\end{aligned}$$

(10pts) (b) Find an equation of the tangent line at $x = 3$.

Solution: We need first to find the slope of the tangent line at $x = 3$. We know that the slope equals the derivative of the function at $x = 3$. Using the formula from the first part we have

$$m = f'(3) = \frac{-1}{2\cdot 4\cdot \sqrt{4}} = -\frac{1}{16}.$$

Moreover $f(3) = 1/2$ so the equation of the tangent line is

$$y - \frac{1}{2} = -\frac{1}{16}(x-3).$$

6. Find the derivative of each of the following functions. Feel free to use the differentiation rules we learned in class.

(10pts) (a)

$$y = \frac{x+2}{x-1}.$$

Solution: We need to use the quotient rule for this problem:

$$\frac{dy}{dx} = \frac{1(x-1)-(x+2)1}{(x-1)^2} = \frac{-3}{(x-1)^2}.$$

(10pts) (b)

$$y = \cos^3(\sqrt{x}).$$

Solution: We need to use the chain rule in order to differentiate this function. First, we can rewrite the function as

$$y = \big(\cos(\sqrt{x})\big)^3.$$

Thus the outside function is $f(x) = x^3$ and the inside function is $g(x) = \cos\sqrt{x}$. Then

$$f'(x) = 3x^2.$$

To compute the derivative of the inside function we use the chain rule one more time. The outside function is $f(x) = \cos x$ and the inside function is $g(x) = \sqrt{x} = x^{1/2}$. Thus

$$\big(\cos\sqrt{x}\big)' = -\sin(\sqrt{x}) \cdot \frac{1}{2}x^{-\frac{1}{2}}$$

and

$$\frac{dy}{dx} = 3\cos^2(\sqrt{x}) \cdot (-\sin(\sqrt{x}) \cdot \frac{1}{2}x^{-\frac{1}{2}}) = -\frac{3}{2}\frac{\cos^2(\sqrt{x})\cdot\sin(\sqrt{x})}{\sqrt{x}}.$$