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1. Let $f(x) = 2x^3 - 3x^2 + 1$.

(5pts) (a) At what values of x does f has a local maximum or a local minimum?

Solution: We find first the critical numbers for the function.

$$f'(x) = 6x^2 - 6x = 6x(x-1).$$

The equation $6x(x-1) = 0$ has two solutions $x = 0$ and $x = 1$. These are the critical points for $f(x)$. We will use the second derivative test to determine whether they are local maximum or local minimum points (you could use the first derivative test as well; however, I do believe that the second derivative test is easier for this problem). The second derivative is

$$f''(x) = 12x - 6.$$

Then $f''(0) = -6 < 0$ and $f''(1) = 6 > 0$. Thus $x = 0$ is a local maximum and $x = 1$ is a local minimum.

(5pts) (b) What are the maximum and minimum values of $f(x)$ on the interval $[-1, 2]$?

Solution: We use the closed interval method to find the absolute maximum and absolute minimum on the given interval. For this we need to evaluate the function at the critical numbers (which we found in part a)) and at the endpoints. We have that $f(0) = 1$, $f(1) = 0$, $f(-1) = -4$, and $f(2) = 5$. Thus the absolute maximum value is 5 (when $x = 2$) and the absolute minimum value is -4 (when $x = -1$).

(5pts) (c) Find the intervals of increase and decrease of f.

Solution: We create the following chart for $f'(x)$:

x			0			1		
$f'(x)$	+	+	0	-	-	0	+	+

Thus the function is increasing on $(-\infty, 0) \bigcup (1, \infty)$ and decreasing on $(0, 1)$.

(5pts) (d) Find the intervals where f is concave up and where is concave down.

Solution: The equation $f''(x) = 0$ is $12x - 6 = 0$ and its solution is $x = 1/2$. We create now the chart for $f''(x)$:

x			1/2		
$f''(x)$	-	-	0	+	+

So $f(x)$ is concave down on $(-\infty, 1/2)$ and concave up on $(1/2, \infty)$.

2. Uranium is a radioactive substance that decays according to an exponential model. It's half-life is 700 milion years (recall that the half-life is the length of time required for the population to be reduced to half its size). Let $y(t)$ be the amount of uranium present at time t, and let y_0 be the original amount.

(5pts) (a) Write the differential equation that describes this model.

Solution: The decay is proportional to the amount of the uranium, so it satisfies the following equation

$$\frac{dy}{dt} = ky.$$

(5pts) (b) Write the solution to this differential equation.

Solution: The solution of such an equation is

$$y(t) = y_0 e^{kt}.$$

We can find the constant k knowiing that the half-life of the uranium is 700 milion years. That means that when $t = 700$ then $y(700) = y_0/2$. Thus

$$\frac{y_0}{2} = y_0 e^{700k}.$$

So $\ln(1/2) = 700k$ and $k = -\ln(2)/700 \simeq -.000099$. Then the solution of the equation is

$$y(t) = y_0 e^{-t\ln(2)/700}.$$

(5pts) (c) If $y_0 = 100$ grams, how much will be left after 1 million years?

Solution: If $y_0 = 100$ and $t = 1$ we obtain

$$y(1) = 100e^{-\ln(2)/700} \simeq 99.901.$$

(10pts) 3. Show that the equation

$$2x - 1 - \sin(x) = 0$$

has exaclty one real solution. (**Hint:** Use the intermediate value theorem and Rolle's theorem)

Solution: Let $f(x) = 2x - 1 - \sin(x)$. Then $f(0) = -1 - \sin(0) = -1 < 0$ and

$$f(\pi/2) = \pi - 1 - \sin(\pi/2) = \pi - 2 > 0.$$

Using the intermediate value theorem we obtain that there must be a numbe a between 0 and $\pi/2$ such that $f(a) = 0$. So our equation has at least one solution. Let's assume that we have two. That is, there are two numbers a and b such that $f(a) = 0 = f(b)$. Rolle's theorem implies than that there is a number c such that $f'(c) = 0$. However

$$f'(x) = 2 - \cos(x) > 0 \text{ for all } x.$$

Thus $f(x)$ can not have two zeros and it has exactly one.

4. Differentiate the following functions:

(5pts) (a)

$$y = x^{\cos x}.$$

Solution: We need to use logarithmic differentiation and use the laws of log to simplify:

$$\ln y = \ln\left(x^{\cos x}\right) = \cos x \ln x.$$

We implicit differentiate the above equation, use the product rule and obtain

$$\frac{1}{y}\frac{dy}{dx} = -\sin x \ln x + \cos x \frac{1}{x}.$$

Thus

$$\frac{dy}{dx} = y\left(-\sin x \ln x + \frac{\cos x}{x}\right) = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$

(5pts) (b)

$$y = \frac{\sqrt{x+1}(2-x)^5}{(x+3)^7}.$$

Solution: There are more than one way to solve this problem. The really hard and long one is to apply the quotient, product and chain rule. The easier one is to use one more time the logarithmic differentiation. We take, first, the natural log in both sides of the equation and we obtain:

$$\begin{aligned} \ln y &= \ln\left(\frac{\sqrt{x+1}(2-x)^5}{(x+3)^7}\right) \\ &= \ln(x+1)^{1/2} + \ln(2-x)^5 - \ln(x+3)^7 \\ &= \frac{1}{2}\ln(x+1) + 5\ln(2-x) - 7\ln(x+3). \end{aligned}$$

Differentiating both sides with respect to x we obtain

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\frac{1}{x+1} - 5\frac{1}{2-x} - 7\frac{1}{x+1}.$$

Thus

$$\frac{dy}{dx} = y\left(\frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+1}\right) = \frac{\sqrt{x+1}(2-x)^5}{(x+3)^7}\left(\frac{1}{2(x+1)} - \frac{5}{2-x} - \frac{7}{x+1}\right).$$

(15pts) 5. A water tank has the shape of an inverted circular cone with base radius 2 m and height 4 m. If water is being pumped into the tank at a rate of 2 m^3/min, find the rate at which the water level is rising when the water is 3 m deep.

Recall that the volume of a cone is given by

$$V = \frac{1}{3}\pi r^2 h.$$

Solution: This problem is Example 3 on page 243 from the textbook. Please read the solution from there.

6. Let $y = \sqrt[3]{x}$ and $a = 27$.

(10pts) (a) Find the linearization $L(x)$ of the function at a and use it to approximate $\sqrt[3]{27.01}$.

Solution: Recall that the linearization of a function is given by

$$L(x) = f(a) + f'(a)(x - a).$$

Here $a = 27$ and $f(3) = \sqrt[3]{27} = 3$. The derivative of the function is

$$f'(x) = \frac{1}{3}x^{-2/3}.$$

Thus $f'(27) = 1/3 \cdot 1/9 = 1/27$. Then the linearization of the function is

$$L(x) = 3 + \frac{1}{27}(x - 27).$$

To approximate $sqrt[3]27.01$ we let $x = 27.01$ in the linearization and we obtain that

$$\sqrt[3]{27.01} \simeq 3 + \frac{1}{27} \cdot 0.01 \simeq 3.00037$$

(10pts) (b) Find the differential dy and evaluate dy for $x = 27$ and $\Delta x = 0.01$.

Solution: The differential of the function is given by

$$dy = f'(x)dx.$$

Here $x = 27$ and $dx = \Delta x = 0.01$. Thus

$$dy = \frac{1}{27} \cdot 0.01 \simeq 0.00037.$$