# Directional Derivatives and the Gradient Vector

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## Directional Derivatives

### Fact
*Recall:*

$$
\begin{aligned}
f_x(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h} \\
f_y(x_0, y_0) &= \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}.
\end{aligned}
$$

## The Directional Derivative

### Definition

- The **directional derivative** of $f$ at $(x_0, y_0)$ in the direction of a **unit** vector $\mathbf{u} = \langle a, b \rangle$ is

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

- If $\mathbf{u} = \mathbf{i} = \langle 1, 0 \rangle$, then $D_{\mathbf{i}} f = f_x$, and if $\mathbf{u} = \mathbf{j} = \langle 0, 1 \rangle$, then $D_{\mathbf{j}} = f_y$.

## The Directional Derivative

Notes

### Theorem

- If $f$ is a differentiable function of $x$ and $y$, then $f$ has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) a + f_y(x, y) b.$$

- If the unit vector $\mathbf{u}$ makes an angle $\theta$ with the positive $x$-axis, then we can write $\mathbf{u} = \langle \cos \theta, \sin \theta \rangle$ and

$$D_{\mathbf{u}} f(x, y) = f_x(x, y) \cos \theta + f_y(x, y) \sin \theta.$$

## Examples

### Examples
- Find the directional derivative of
$$f(x,y) = x^3 - 3xy + 4y^2$$
  at the point $(1,2)$ in the direction $\theta = \pi/6$.
- Find the directional derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction of $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j}$.

## The Gradient Vector

### Definition
- If $f$ is a function of two variables $x$ and $y$, then the **gradient** of $f$ is the vector function $\nabla f$ defined by

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$

# Example

### Example
- Find the gradient of $f(x, y) = \sin x + e^{xy}$ at $(0, 1)$.

### Fact
- *The equation of the directional derivative becomes:*

$$D_{\mathbf{u}}f(x, y) = \nabla f(x, y) \cdot \mathbf{u}.$$

## Examples

### Example

Find the directional derivative of the function $f(x, y) = x^2 y^3 - 4y$ at the point $(2, -1)$ in the direction of the vector $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j}$.

## Functions of three variables

### Definition

- If $w = f(x, y, z)$ is a function of three variables, the **directional derivative** of $f$ at $(x_0, y_0, z_0)$ in the direction of the unit vector $\langle a, b, c \rangle$ is

$$D_u f(x_0, y_0, z_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb, z_0 + hc) - f(x_0, y_0, z_0)}{h}$$

  if the limit exists.
- Then

$$D_u f(x, y, z) = f_x(x, y, z)a + f_y(x, y, z)b + f_z(x, y, z)c.$$

## Functions of three variables

### Definition

- The gradient is
$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

- The formula for the directional derivative become
$$D_u f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}.$$

## Examples

### Examples

Consider the function $f(x, y, z) = xy^2 + yz^3 + xy^2$.

- Find the gradient of $f$.
- Find the gradient of $f$ at the point $(5, 4, -1)$.
- Find the rate of change of the function $f$ at the point $(5, 4, -1)$ in the direction of $\mathbf{u} = \langle 2/\sqrt{22}, -3/\sqrt{22}, -3/\sqrt{22} \rangle$.