15.1: Double Integrals over Rectangles

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Volumes and Double Integrals

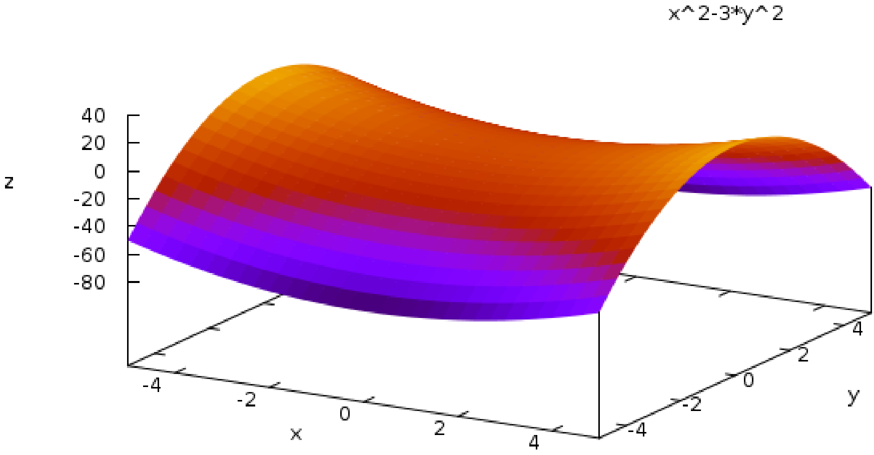
- Let f be a function of two variables defined on a closed rectangle:

$$R = \{(x, y) \in \mathbb{R}^2 \ : \ a \leq x \leq b, c \leq y \leq d\}$$

- The graph of f is a surface with equation $z = f(x, y)$.
- The solid S that lies above R and under the graph of f is

$$S = \{(x, y, z) \in \mathbb{R}^3 \ : \ 0 \leq z \leq f(x, y), \ (x, y) \in \mathbb{R}\}$$

Example



The Volume

- To find the volume of S we divide the rectangle R into subrectangles

$$R_{ij} = \{(x, y), : \ x_i \leq x \leq x_{i+1}, y_j \leq y \leq y_{j+1}\}.$$

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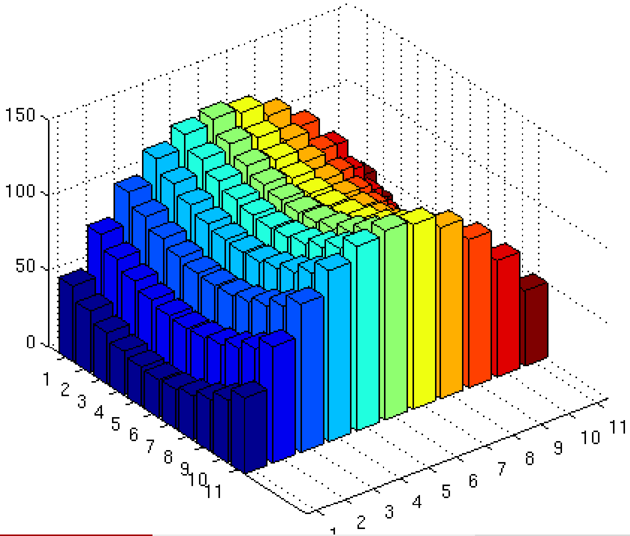
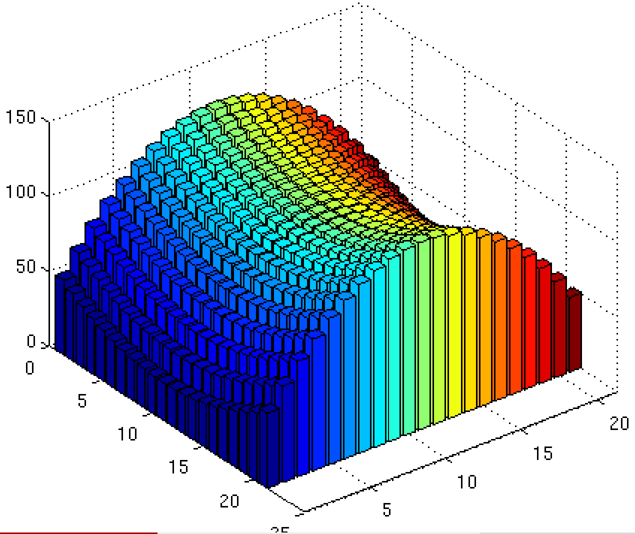
- Choose a **sample point** (x_{ij}^*, y_{ij}^*) in each R_{ij}
- The volume of S is approximated by

$$V \simeq \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$

Example



Example (cont'd)



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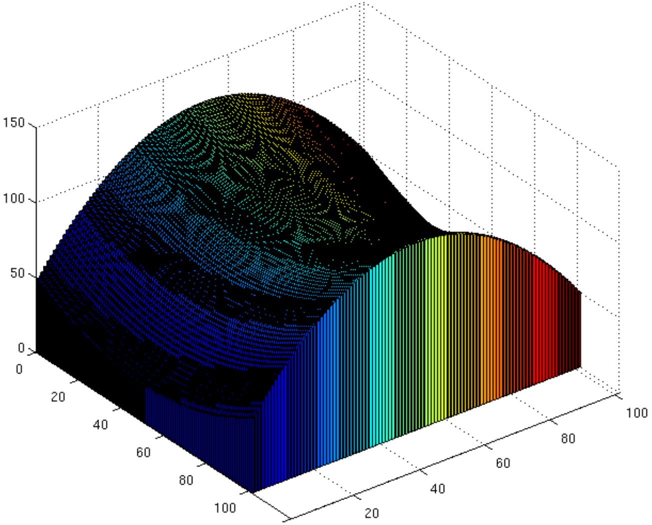

The Double Integral

Definition

The **double integral** of f over the rectangle R is

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$

Volume = Double Integral

Fact

If $f(x, y) \geq 0$, *then the volume* V *of the solid that lies above the rectangle* R *and below the surface* $z = f(x, y)$ *is*

$$V = \iint_R f(x, y) dA.$$

Example

Example

Estimate $\iint_R (1 - xy^2) dA$, where $R = [0, 4] \times [-1, 2]$; use a Riemann sum with $m = 2$ and $n = 3$.

Midpoint Rule for Double Integrals

Fact

$$\iint_r f(x,y)dA \simeq \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i,\overline{y}_i)\Delta A,$$

where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_i$ is the midpoint of $[y_{i-1}, y_i]$.

Example

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Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)dA$, where $R = \{(x, y) : 0 \le x \le 2, 1 \le y \le 2\}$.