# 15.1: Double Integrals over Rectangles

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Notes

## Volumes and Double Integrals

- Let $f$ be a function of two variables defined on a closed rectangle:

$$R = \{(x, y) \in \mathbb{R}^2 \ : \ a \leq x \leq b, c \leq y \leq d\}$$

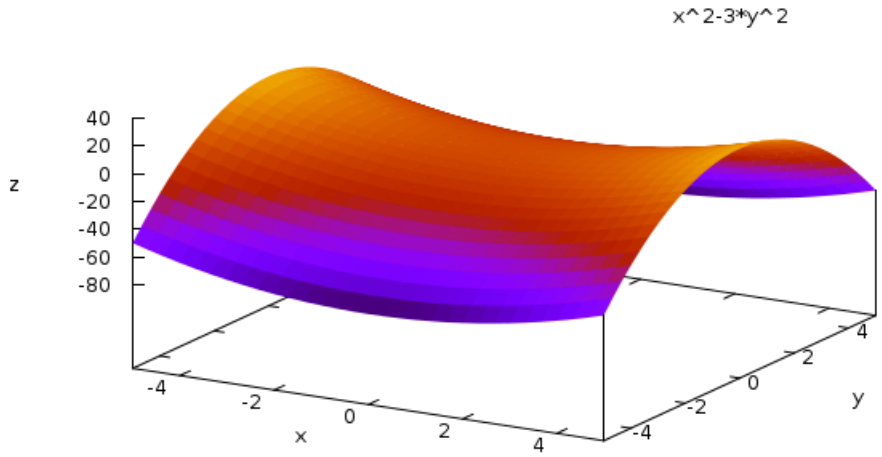
- The graph of $f$ is a surface with equation $z = f(x, y)$.
- The solid $S$ that lies above $R$ and under the graph of $f$ is

$$S = \{(x, y, z) \in \mathbb{R}^3 \ : \ 0 \leq z \leq f(x, y), (x, y) \in \mathbb{R}\}$$



Notes

## Example






Notes

## The Volume

- To find the volume of $S$ we divide the rectangle $R$ into subrectangles

$$R_{ij} = \{(x, y), : x_i \leq x \leq x_{i+1}, y_j \leq y \leq y_{j+1}\}.$$

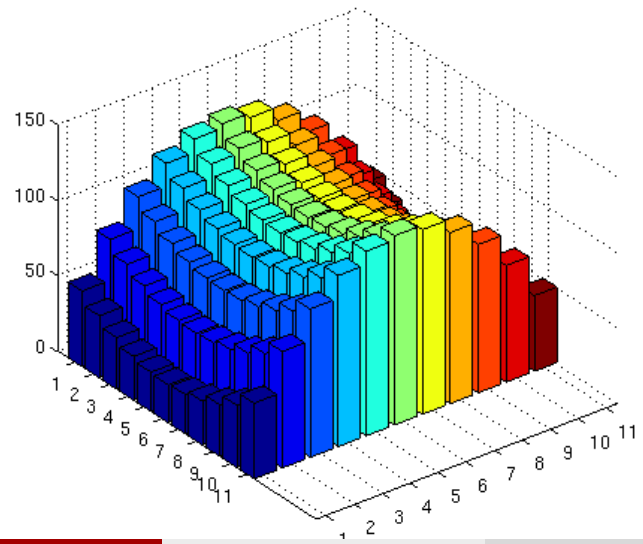
- Choose a **sample point** $(x_{ij}^*, y_{ij}^*)$ in each $R_{ij}$
- The volume of $S$ is approximated by

$$V \simeq \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*) \Delta A.$$



Notes

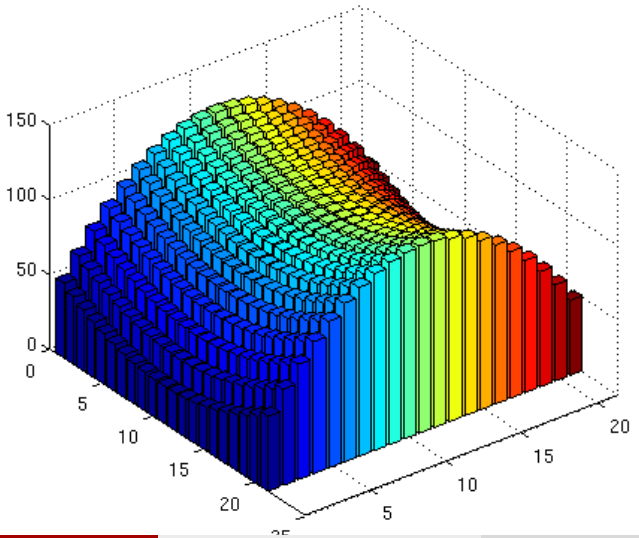
## Example






Notes

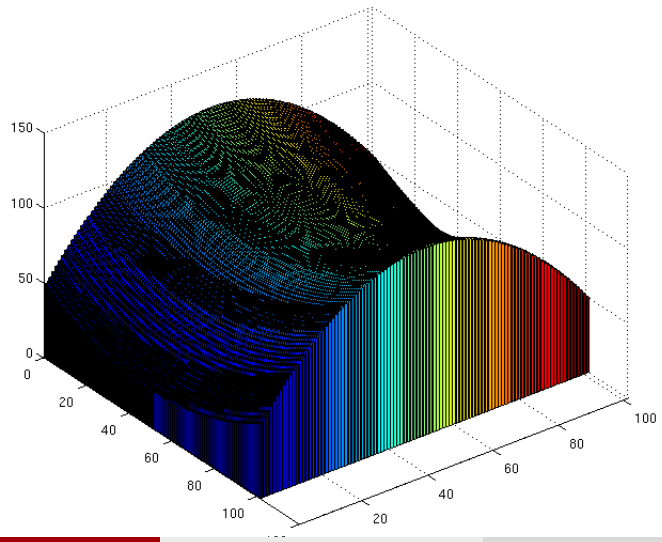
## Example (cont'd)






Notes

## Example (cont'd)






Notes

## The Double Integral

**Definition**

The **double integral** of $f$ over the rectangle $R$ is

$$\iint_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$



Notes

## Volume = Double Integral

**Fact**

*If $f(x,y) \geq 0$, then the volume $V$ of the solid that lies above the rectangle $R$ and below the surface $z = f(x,y)$ is*

$$V = \iint_R f(x,y)dA.$$



Notes

## Example

**Example**

Estimate $\iint_R (1 - xy^2)dA$, where $R = [0,4] \times [-1,2]$; use a Riemann sum with $m = 2$ and $n = 3$.



Notes

## Midpoint Rule for Double Integrals

**Fact**

$$\iint_r f(x,y)dA \simeq \sum_{i=1}^{m}\sum_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\Delta A,$$

*where $\overline{x}_i$ is the midpoint of $[x_{i-1}, x_i]$ and $\overline{y}_j$ is the midpoint of $[y_{i-1}, y_i]$.*

Notes



## Example

**Example**

Use the Midpoint Rule with $m = n = 2$ to estimate the value of the integral $\iint_R (x - 3y^2)dA$, where $R = \{(x,y) : 0 \leq x \leq 2, 1 \leq y \leq 2\}$.

Notes