15.3: Double Integrals over General Regions

Marius Ionescu

November 19, 2012
Fact

We want to integrate a function $f$ over bounded regions $D$ of more general shape:
Definition

If $D$ is a bounded region, then we define a new function $F$ with domain a rectangle $R$ that contains $D$ by

$$F(x, y) = \begin{cases} f(x, y) & \text{if } (x, y) \text{ is in } D \\ 0 & \text{otherwise} \end{cases}$$
Definition
The **double integral of $f$ over $D$** is

$$
\iint_D f(x, y) \, dA = \iint_R F(x, y) \, dA.
$$
Domains of type I

Definition
A domain $D$ is of type I if it lies between the graphs of two continuous functions of $x$:

$$D = \{(x, y) \mid a \leq x \leq b, \ g_1(x) \leq y \leq g_2(x)\}$$
Fact

If $D$ is a region of type I and $f$ is continuous then

$$\int \int_D f(x, y) \, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y) \, dy \, dx.$$
Examples

Evaluate the following double integrals:

\[ \int_D \frac{1}{x^5 + 1} \, dA \], where 
\[ D = \{ (x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x^2 \} \]

\[ \int_D (x + 2y) \, dA \], where 
\[ D \text{ is bounded by the parabolas } y = 2x^2 \text{ and } y = 1 + x^2. \]

\[ \int_D (x^2 + 2y) \, dA \], where 
\[ D \text{ is bounded by } y = x, y = x^2, x \geq 0. \]
Examples

Evaluate the following double integrals:

- \( \iint_D \frac{y}{x^5 + 1} \, dA \), where \( D = \{(x, y) \mid 0 \leq 1, 0 \leq y \leq x^2\} \)
Examples

Evaluate the following double integrals:

1. \( \iint_D \frac{y}{x^5+1} \, dA \), where \( D = \{ (x, y) \mid 0 \leq 1, 0 \leq y \leq x^2 \} \)

2. \( \iint_D (x + 2y) \, dA \), where \( D \) is the region bounded by the parabolas \( y = 2x^2 \) and \( y = 1 + x^2 \).
Examples

Evaluate the following double integrals:

1. \( \iint_D \frac{y}{x^2+1} \, dA \), where \( D = \{(x, y) \mid 0 \leq 1, 0 \leq y \leq x^2\} \)
2. \( \iint_D (x + 2y) \, dA \), where \( D \) is the region bounded by the parabolas \( y = 2x^2 \) and \( y = 1 + x^2 \).
3. \( \iint_D (x^2 + 2y) \, dA \), where \( D \) is bounded by \( y = x \), \( y = x^2 \), \( x \geq 0 \).
Domains of type II

Definition
A domain $D$ is of type II if it can be expressed as

$$D = \{(x, y) : c \leq y \leq d, \ h_1(y) \leq x \leq h_2(y)\}.$$
Fact

If $D$ is a region of type II and $f$ is continuous then

$$
\iint_D f(x, y) \, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y) \, dx \, dy.
$$


Examples

Evaluate the following integrals

\[ \iint_D xy \, dA \], where \( D \) is the region bounded by the line \( y = x - 1 \) and the parabola \( y^2 = 2x + 6 \).

\[ \iint_D y^2 e^{xy} \, dA \], where \( D \) is the region bounded by \( y = x \), \( y = 4 \), \( x = 0 \).
Examples

Evaluate the following integrals

- \( \iint_D xy \, dA \), where \( D \) is the region bounded by the line \( y = x - 1 \) and the parabola \( y^2 = 2x + 6 \).
Examples

Evaluate the following integrals

1. \( \iint_D xy \, dA \), where \( D \) is the region bounded by the line \( y = x - 1 \) and the parabola \( y^2 = 2x + 6 \).

2. \( \iint_D y^2 e^{xy} \, dA \), where \( D \) is the region bounded by \( y = x \), \( y = 4 \), \( x = 0 \).
More Examples

Examples

Evaluate the iterated integral

$$\int_{0}^{1} \int_{x}^{y^2} \sin(y^2) \, dy \, dx$$

Find the volume of the tetrahedron bounded by the planes

$$x + 2y + z = 2, \quad x = 2y, \quad x = 0, \quad z = 0.$$
More Examples

Examples

- Evaluate the iterated integral \( \int_0^1 \int_x^1 \sin(y^2) \, dy \, dx \).
More Examples

Examples

- Evaluate the iterated integral \( \int_0^1 \int_x^1 \sin(y^2) \, dy \, dx \).
- Find the volume of the tetrahedron bounded by the planes \( x + 2y + z = 2, \ x = 2y, \ x = 0, \) and \( z = 0 \).
More Examples

Examples

- Evaluate the iterated integral \( \int_0^1 \int_x^1 \sin(y^2) \, dy \, dx \).
- Find the volume of the tetrahedron bounded by the planes \( x + 2y + z = 2 \), \( x = 2y \), \( x = 0 \), and \( z = 0 \).
- Find the volume of the solid under the surface \( z = 1 + x^2 y^2 \) and above the region enclosed by \( x = y^2 \) and \( x = 4 \).
Properties of the double integral

Fact

Properties of the double integral:

- $\iint_D [f(x, y) + g(x, y)]dA = \iint_D f(x, y)dA + \iint_D g(x, y)dA$.
- $\iint_D cf(x, y)dA = c \iint_D f(x, y)dA$.
- If $f(x, y) \geq g(x, y)$ for all $(x, y)$ in $D$, then $\iint_D f(x, y)dA \geq \iint_D g(x, y)dA$.
- If $D = D_1 \cup D_2$, where $D_1$ and $D_2$ don’t overlap except perhaps on their boundaries, then $\iint_D f(x, y)dA = \iint_{D_1} f(x, y)dA + \iint_{D_2} f(x, y)dA$. 

Properties of the double integral (cont’d)

Fact

Properties of the double integral:

- \[ \iint_D 1 \, dA = \text{area of } D = A(D). \]
- If \( m \leq f(x, y) \leq M \), then \( mA(D) \leq \iint_D f(x, y) \, dA \leq MA(D) \).
Examples

Examples
Examples

Find the area of the triangle with vertices \((0, 0), (5, 0),\) and \((5, 4)\) (using double integrals).
Examples

- Find the area of the triangle with vertices (0, 0), (5, 0), and (5, 4) (using double integrals).
- Estimate the integral \( \iint_D e^{\sin x \cos y} dA \), where \( D \) is the disk with center the origin and radius 2.