15.4: Double Integrals in Polar Coordinates

Marius Ionescu

12/3/2012

Polar Coordinates

Fact

The polar coordinates (r, θ) *of a point are related to the rectangular coordinates* (x, y) *by the equations*

$$\begin{aligned} r^2 &= x^2 + y^2 \\ x &= r\cos\theta \\ y &= r\sin\theta. \end{aligned}$$

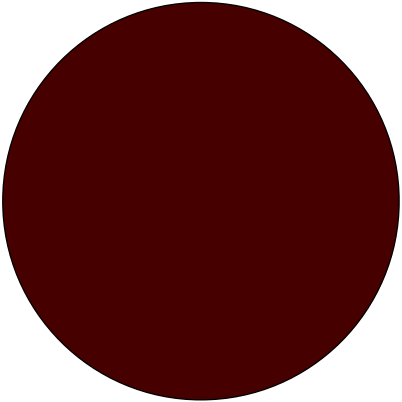
Polar Rectangles

Definition

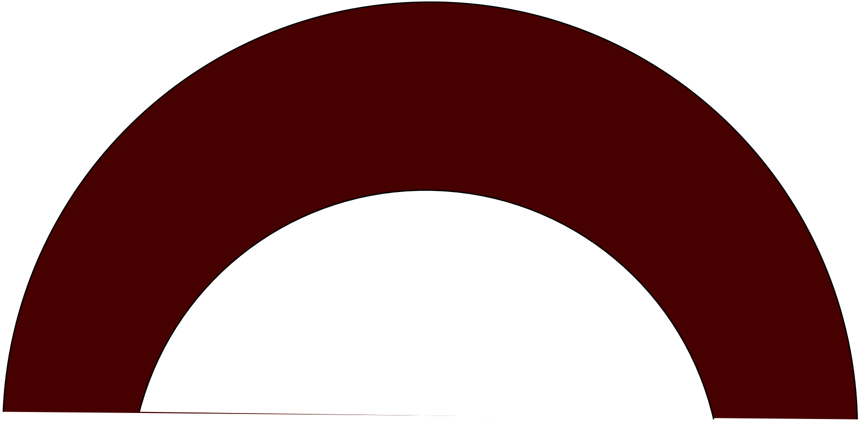
A **polar rectangle** is a domain that can be expressed as

$$D = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}.$$

Examples of polar rectangles



Another example



Change to Polar Coordinates in a Double Integral

Fact

If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x, y) dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta) r \, dr d\theta.$$

Examples

Examples

Evaluate the following integrals:

Examples

Examples

Evaluate the following integrals:

- $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Examples

Examples

Evaluate the following integrals:

- $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
- $\iint_R (2x - y)\,dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.

Examples

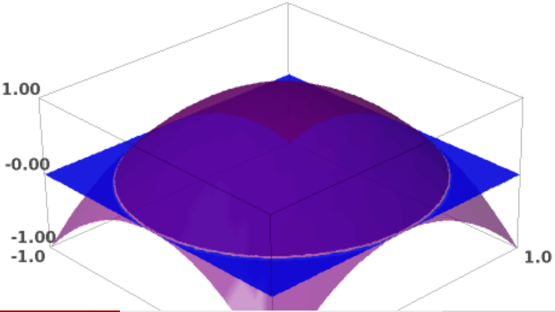
Evaluate the following integrals:

- $\iint_R (3x + 4y^2)dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
- $\iint_R (2x - y)dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.
- $\iint_D e^{-x^2-y^2}dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.

Example

Example

Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.



Type II domains in polar coordinates

Fact

If f is continuous on a polar region of the form

$$D = \{(r,\theta) \,:\, \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$$

then

$$\iint_D f(x,y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta) r dr d\theta.$$

Examples

Examples

Examples

Examples

- Evaluate $\iint_D x\,dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.

Examples

Examples

- Evaluate $\iint_D x\,dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.
- Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.