15.4: Double Integrals in Polar Coordinates

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Polar Coordinates

Fact

The polar coordinates (r, θ) of a point are related to the rectangular coordinates (x, y) by the equations

$$
\begin{aligned}
r^2 &= x^2 + y^2 \\
x &= r \cos \theta \\
y &= r \sin \theta.
\end{aligned}
$$

Polar Rectangles

Definition

A **polar rectangle** is a domain that can be expressed as

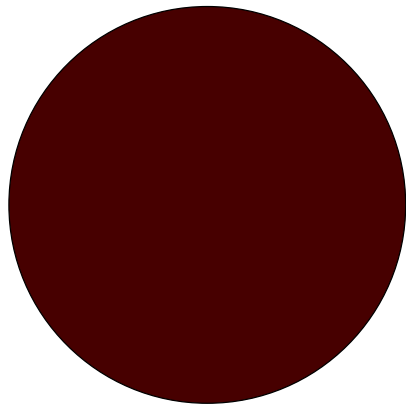$$D = \{(r, \theta) \mid a \leq r \leq b, \alpha \leq \theta \leq \beta\}.$$

Examples of polar rectangles

Another example
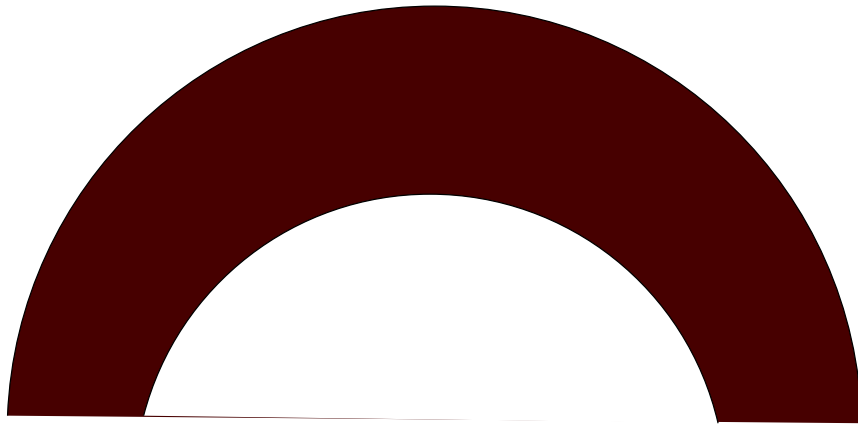
Change to Polar Coordinates in a Double Integral

Fact

If f is continuous on a polar rectangle R given by $0 \leq a \leq r \leq b$, $\alpha \leq \theta \leq \beta$, where $0 \leq \beta - \alpha \leq 2\pi$, then

$$\iint_R f(x,y)dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)r\,dr\,d\theta.$$

Examples

Examples

Evaluate the following integrals:

- $\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.
- $\iint_R (2x - y)\,dA$, where R is the region in the first quadrant enclosed by the circle $x^2 + y^2 = 4$ and the lines $x = 0$ and $y = x$.
- $\iint_D e^{-x^2-y^2}\,dA$, where D is the region bounded by the semicircle $x = \sqrt{4 - y^2}$ and the y-axis.

Example

Example

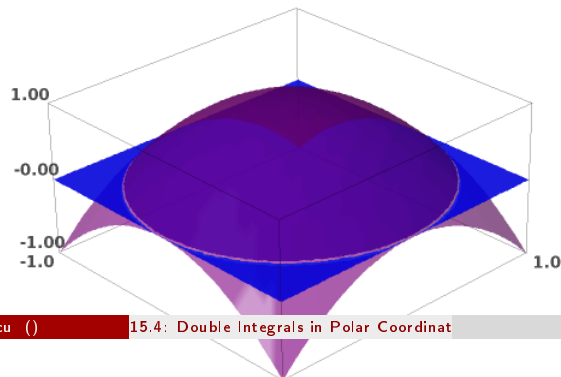Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Type II domains in polar coordinates

Fact

If f is continuous on a polar region of the form

$$D = \{(r, \theta) \ : \ \alpha \leq \theta \leq \beta, \ h_1(\theta) \leq r \leq h_2(\theta)\}$$

then

$$\iint_D f(x, y)dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta.$$

Examples

Examples

- Evaluate $\iint_D x\, dA$, where D is the region in the first quadrant that lies between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2x$.
- Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos 2\theta$.