

The Dot Product

Lecture 2

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The Dot Product

- If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of \mathbf{a} and \mathbf{b} is the number

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3.$$

- It is also called the **scalar product** or **inner product**.

Examples

- $\langle 2, 1 \rangle \cdot \langle -1, 3 \rangle$.
- $\langle 3, -2, 1 \rangle \cdot \langle 0, 1, 1 \rangle$.

Properties of the Dot Product

- If \mathbf{a} , \mathbf{b} , and \mathbf{c} are vectors in V_3 and c is a scalar, then

① $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$

② $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$

③ $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

④ $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$

⑤ $\mathbf{0} \cdot \mathbf{a} = 0.$

The angle between two vectors

- If θ is the angle between the vectors \mathbf{a} and \mathbf{b} , then

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

- If θ is the angle between the nonzero vectors \mathbf{a} and \mathbf{b} , then

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Orthogonal vectors

- \mathbf{a} and \mathbf{b} are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Direction Angles

- The **direction angles** of a nonzero vector \mathbf{a} are the angles α , β , and γ that \mathbf{a} makes with the positive x -, y -, and z -axes.
- The cosines of these direction angles are called the **direction cosines** of the the vector \mathbf{a} :

$$\cos \alpha = \frac{a_1}{|\mathbf{a}|}, \quad \cos \beta = \frac{a_2}{|\mathbf{a}|}, \quad \cos \gamma = \frac{a_3}{|\mathbf{a}|}.$$

Projections

- Scalar projection of \mathbf{b} onto \mathbf{a} :

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|}$$

- Vector projection of \mathbf{b} onto \mathbf{a}

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$$

Work done by a constant force

- The work done by a constant force \mathbf{F} is

$$\mathbf{F} \cdot \mathbf{D},$$

where \mathbf{D} is the displacement vector.

- Example: A constant force $\mathbf{F} = -2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$ moves an object along a straight line from the point $(1, 0, 0)$ to $(-3, 2, 3)$. Find the work done.