12.5: Lines and Planes in $\mathbb{R}^3$

Lecture 4

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The vector equation of a line

- A line L is determined when we know a point $P_0(x_0, y_0, z_0)$ on L and the direction of L.
- Let v be a vector parallel to L and let r_0 the position vector of P_0
- The **vector equation** of L is

$$\mathbf{r} = \mathbf{r_0} + t\mathbf{v},$$

where t is the **parameter.**

Examples

- Find the vector equation for the line trough the point $P(-1, 2, 2)$ and parallel to the vector $\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$.
- Find the vector equation for the line trough the point $P(1, -1, 2)$ and parallel to the vector $\langle 2, 0, -3 \rangle$.

The parametric equations of a line

- If $\mathbf{r} = \langle x, y, z \rangle$, $\mathbf{v} = \langle a, b, c \rangle$, and $\mathbf{r}_0 = \langle x_0, y_0, z_0 \rangle$, then

$$\langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

- The **parametric equations:**

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

Examples

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The line segment between two points

- The line segment from $\mathbf{r}_0$ and $\mathbf{r}_1$ is given by the vector equation
$$\mathbf{r}(t) = (1 - t)\mathbf{r}_0 + t\mathbf{r}_1 \quad 0 \leq t \leq 1$$

Examples

- Find vector and parametric equations for the line trough $(4, 1, 0)$ that is parallel to the line with parametric equations $x = 1 + 2t, y = 2 + 3t, z = 1 - t$.
- Find the point of intersection of this new line with each of the coordinate planes.

Planes

- A plane is determined by a point $P_0(x_0, y_0, z_0)$ in the plane and a vector $\mathbf{n}$ that is orthogonal to the plane.
- The vector $\mathbf{n}$ is called a **normal vector.**
- If $\mathbf{r}_0$ is t he position vector of P_0 and $\mathbf{r}$ then the **vector equation of the plane**

$$\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$$

The Scalar equation of a plane

- The scalar equation of the plane through $P_0(x_0, y_0, z_0)$ with normal vector $n = \langle a, b, c \rangle$ is
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$
- Example: Find an equation of the plane through the point $(2, 1, -3)$ with normal vector $\mathbf{n} = \langle 3, 1, 1 \rangle$

The linear equation of a plane

- By collecting the terms

$$ax + by + cz + d = 0$$

- Example: Find an equation of a plane through the point $(-2, -1, 2)$ which is parallel to the plane $-3x + 2y + z = 7$.

The angle between two planes

- Two planes are parallel if their normal vectors are parallel.
- If two planes are not parallel, then they intersect in a straight line and the angle between them is the (acute) angle between their normal vectors.

Examples

1. Find the angle between the planes $x - y + z = 1$ and $2x + y - 3 = 1$.
2. Find a equation for the line of intersection L of these two planes

More Examples

- Find a equation of a plane containing the three points $(-2, 2, 0)$, $(-1, 3, 1)$ and $(-3, -3, 2)$
- Find an equation of a plane through the point $(0, 4, 1)$ which is orthogonal to the line $x = 1 + t$, $y = 2 - 3t$, $z = 5 + 2t$ in which the coefficient of x is 5.
- Find an equation of a plane containing the line $\mathbf{r} = \langle -2, -2, 1\rangle + t\langle -4, 0, 1\rangle$ which is parallel to the plane $-2x + 2y + z = 5$