Sections 13.1 and 13.2: Vector Functions, Space Curves, and Derivatives

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Vector Functions

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- We write

$$\begin{aligned} \mathbf{r}(t) &= \langle f(t), g(t), h(t) \rangle \\ &= f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \end{aligned}$$

Limit of a vector function

Fact

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, *then*

$$\lim_{t\to a} \mathbf{r}(t) = \langle \lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t) \rangle$$

Example

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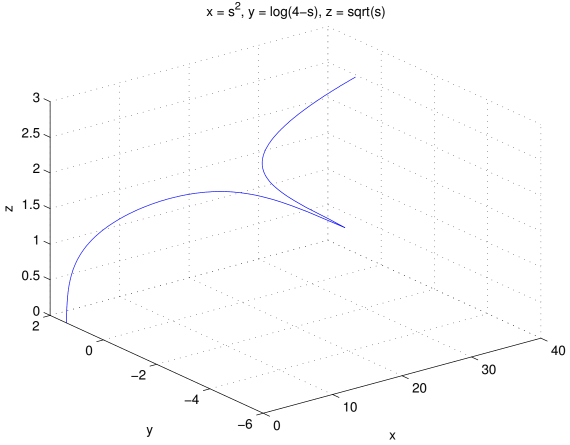
Find the limit

$$\lim_{t \to \pi/4} \langle \cos t, \sin t, t \rangle$$

Space Curves

Example

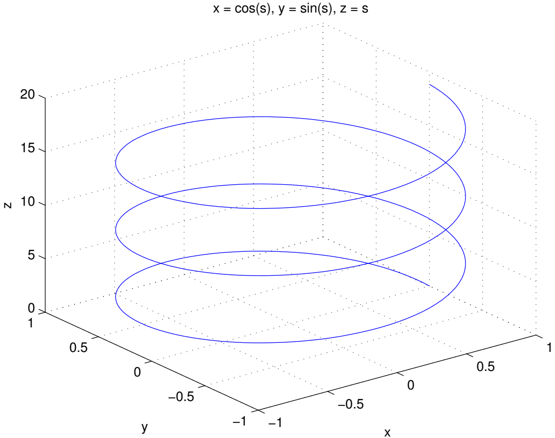
$\mathbf{r}(t) = \langle t^2, \ln(4-t), \sqrt{t} \rangle$



The Helix

Example

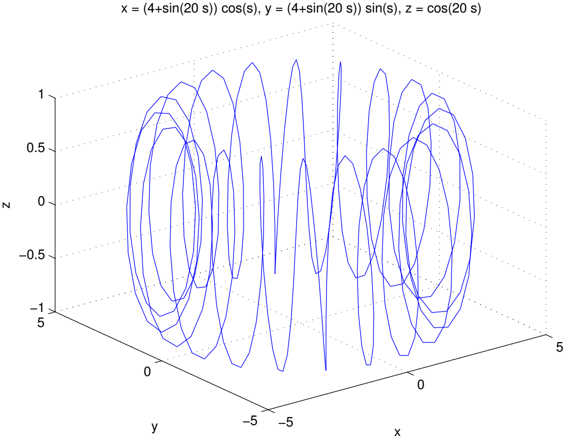
$r(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$



The Toroidal

Example

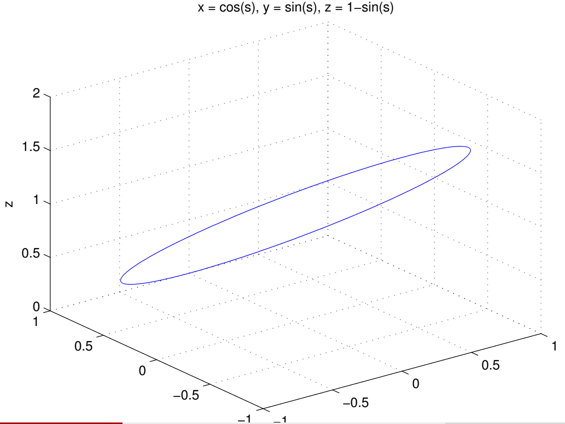
$\mathbf{r}(t) = (4 + \sin(2t))\cos t\mathbf{i} + (4 + \sin 20t)\sin t\mathbf{j} + \cos 20t\mathbf{k}$



Example

Example

Find a vector function that represents the curve of intersection of the cylinder $x^2 + y^2 = 1$ and the plane $y + z = 1$.



Derivatives of Vector Functions

- The derivative $\mathbf{r}'$ of $\mathbf{r}$ is

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

if the limit exists.

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- The vector $\mathbf{r}'(t)$ is called the **tangent vector**.
- The **unit tangent vector** is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

How to compute $\mathbf{r}'(t)$?

- If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, where f, g, and h are differentiable functions, then
$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle$$

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- Find the derivative of $\mathbf{r}(t) = (2t + t^2)\mathbf{i} + e^{-t^2}\mathbf{j} + \sin(2t)\mathbf{k}$.

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- Find the derivative of $\mathbf{r}(t) = (2t + t^2)\mathbf{i} + e^{-t^2}\mathbf{j} + \sin(2t)\mathbf{k}$.
- Find parametric equations for the tangent line to the helix with parametric equations

$$x = 2\cos t,\ y = \sin t,\ z = t$$

at the point $(0, 1, \pi/2)$.

Differentiation Rules

$$
\begin{aligned}
[\mathbf{u}(t)+\mathbf{v}(t)]' &= \mathbf{u}'(t)+\mathbf{v}'(t)\\
(c\mathbf{u}(t))' &= c\mathbf{u}'(t)\\
(f(t)\mathbf{u}(t))' &= f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)\\
(\mathbf{u}(t)\cdot\mathbf{v}(t))' &= \mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)\\
(\mathbf{u}(t)\times\mathbf{v}(t))' &= \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)\\
(\mathbf{u}(f(t)))' &= f'(t)\mathbf{u}'(f(t)).
\end{aligned}
$$