13.3 and 13.4: Arc Length, Curvature and Motion in Space

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Recall: the Length of a Plane Curve

Fact

Suppose that a plane curve has parametric equations $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$. The length is given by the formula

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt. \end{aligned}$$

The Length of A Space Curve

Definition

- The length of a space curve $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ is

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt$$
$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt.$$

- Note that

$$L = \int_a^b |\mathbf{r}'(t)| dt.$$

Example

Example

- Find the length of the arc of the circular helix with vector equation $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ from the point $(1, 0, 0)$ to the point $(1, 0, 2\pi)$.
- Find the point where you have to stop if you want to travel only half of the previous length.

Example

Example

Find the length of the curve $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \ln t\mathbf{k}$ with $0 \le t \le 1$.

The arc length function

Definition

The **arc length function** s is

$$s(t) = \int_a^t |r'(u)| du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2} du.$$

Example

Example

Reparametrize the helix $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}$ with respect to arc length measure from $(1, 0, 0)$ in the direction of increasing t.

Curvature

Definition

1. A parametrization $\mathbf{r}(t)$ is smooth if it has no sharp corners or cups; formally: $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq 0$.
2. The **curvature** of a curve is

$$k = \left| \frac{d\mathbf{T}}{ds} \right|,$$

where $\mathbf{T}$ is the unit tangent vector.

Useful formulas

Fact

1. $$k(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

2. $$k(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|r'(t)|^3}$$

Examples

Example

Find the curvature of:

1. $\mathbf{r}(t) = \langle t^2, \ln t, t \ln t\rangle$ at the point $(1, 0, 0)$.
2. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^t\mathbf{k}$.

Motion in Space: Velocity and Acceleration

Definition

- Let $\mathbf{r}(t)$ be a space curve. The **velocity vector** $\mathbf{v}(t)$ at time t is
$$\mathbf{v}(t) = \lim_{h \to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \mathbf{r}'(t).$$
- The **speed** of the particle at time t is the magnitude of the velocity, that is, $|\mathbf{v}(t)|$.
- The **acceleration** of the particle is defined as the derivative of the velocity:
$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t).$$

Examples

Example

- Find the velocity, acceleration, and speed of a particle with position vector $\mathbf{r}(t) = \langle t^2, e^t, te^t \rangle$.
- Find the velocity and position vectors of a particle that has the acceleration $\mathbf{a}(t) = -10\mathbf{k}$, initial velocity $\mathbf{v}(0) = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and initial position $\mathbf{r}(0) = 2\mathbf{i} + 3\mathbf{j}$.

More Examples

Example

A projectile is fired with an angle of elevation of $\pi/6$ and initial velocity of 1200km/sec. Where does the projectile hit the ground and with what speed?

- **Recall:** Newton's Second Law of Motion

$$\mathbf{F}(t) = m\mathbf{a}(t).$$

- So

$$\mathbf{F}(t) = m\mathbf{a} = -mg\mathbf{j},$$

  where $g = |\mathbf{a}| \simeq 9.8m/s^2$.