1. A parallelogram has one vertex $P = (1, 0, 0)$ and two adjacent vertices $Q = (0, 1, 1)$ and $R = (2, 2, 1)$.

(10pts) (a) Find the area of the parallelogram.

Solution: Recall that the area of the parallelogram determined by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is given by the length of the cross product: Area $= |\overrightarrow{PQ} \times \overrightarrow{PR}|$. We have $\overrightarrow{PQ} = \langle -1, 1, 1\rangle$ and $\overrightarrow{PR} = \langle 1, 2, 1\rangle$. Then $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle -1, 2, -3\rangle$. Hence the area is given by its length $\sqrt{1+4+9} = \sqrt{14}$.

(10pts) (b) Find the cosine of the angle between the side PQ and the diagonal from P to the opposite vertex (this is **not** the angle between PQ and PR).

Solution: Let S be the opposite vertex. Then $\overrightarrow{PS} = \overrightarrow{PQ} + \overrightarrow{PR} = \langle 0, 3, 2\rangle$. The angle between $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is given by the formula

$$\cos\theta = \frac{|\overrightarrow{PQ} \cdot \overrightarrow{PS}|}{|\overrightarrow{PQ}||\overrightarrow{PS}|}.$$

We have that $\overrightarrow{PQ} \cdot \overrightarrow{PS} = \langle 0, 3, 2\rangle \cdot \langle -1, 1, 1\rangle = 5$, $|\overrightarrow{PQ}| = \sqrt{3}$, and $|\overrightarrow{PS}| = \sqrt{13}$. Hence

$$\cos\theta = \frac{5}{\sqrt{3}\sqrt{13}}.$$

(10pts) 2. (a) Find the parametric equation of the straight line passing through $(6, 3, 7)$ and perpendicular to the plane $2x - y + 3z = 2$.

Solution: Since the line is perpendicular to the plane, the direction of the line is given by the normal to the plane: $\mathbf{n} = \langle 2, -1, 3 \rangle$. Then the parametric equation is $x = 6 + 2t,\ y = 3 - t,\ z = 7 + 3t$.

(10pts) (b) What is the point of intersection of the line in (a) and the plane in (a)?

Solution: The point must satisfy both the equation of the line and the equation of the plane. Using the formulas from the previous part we have

$$2(6 + 2t) - (3 - t) + 3(7 + 3t) = 2.$$

Solving this equation for t we obtain that $t = -2$. Plugging this value into the parametric equations we have that $x = 2,\ y = 5$, and $z = 1$. Hence the point of intersection is $(2, 5, 1)$.

3. The position vector of an object moving in space is given by

$$\mathbf{r}(t) = 3t^2\mathbf{i} + 12t\mathbf{j} + 8t^{3/2}\mathbf{k}$$

at time t.

(10pts) (a) What is the speed of the object at time $t = 2$?

> **Solution:** The speed is the length of the velocity vector. Then $\mathbf{v} = \mathbf{r}'(t) = \langle 6t, 12, 12t^{1/2}\rangle$. So $\mathbf{v}(2) = \langle 12, 12, 12\sqrt{2}\rangle$ and $|\mathbf{v}(2)| = \sqrt{12^2 + 12^2 + 12^2 \cdot 2} = 24$.

(10pts) (b) How far has the object travelled along the curve in going from $(0, 0, 0)$ to $(48, 48, 64)$? (Note: You are not asked to find the distance between $(0, 0, 0)$ and $(48, 48, 64)$.)

> **Solution:** We have to compute the length of the arc starting at $(0, 0, 0)$ and ending at $(48, 48, 64)$. Using the formula for $\mathbf{r}(t)$, this means from $t = 0$ to $t = 4$. Hence we have to compute the integral
>
> $$\begin{aligned} L &= \int_0^4 |\mathbf{r}'(\mathbf{t})|\mathrm{d}t = \int_0^4 \sqrt{36t^2 + 12^2 + 12^2 t}\,\mathrm{d}t \\ &= \int_0^4 \sqrt{36(t^2 + 4t + 4)}\,\mathrm{d}t = 6\int_0^4 \sqrt{(t+2)^2}\,\mathrm{d}t \\ &= = 6\int_0^4 (t+2)\mathrm{d}t = 6\left(\frac{t^2}{2} + 2t\right)\Bigg|_0^4 = 96. \end{aligned}$$

(20pts) 4. Find the velocity and position vector of a particle that has the acceleration $\vec{a}(t) = e^t\vec{j}$, initial velocity $\vec{v}(0) = \vec{i} - \vec{k}$ and initial position vector $\vec{r}(0) = 2\vec{j}$.

Solution: We have that $\vec{v}(t) = \int \vec{a}(t)dt = e^t\vec{j} + \vec{C}$. Since $\vec{v}(0) = \vec{i} - \vec{j}$, it follows that $\vec{C} = \vec{i} - \vec{j} - \vec{k}$. Thus $\vec{v}(t) = \vec{i} + (e^t - 1)\vec{j} - \vec{k}$. To find the position vector of the particle we need to integrate $\vec{v}(t)$. Thus

$$\vec{r}(t) = \int \vec{v}(t)dt = t\vec{i} + (e^t - t)\vec{j} - t\vec{k} + \vec{C}.$$

Since $\vec{r}(0) = 2\vec{j}$, it follows that $\vec{C} = \vec{j}$. So $\vec{r}(t) = t\vec{i} + (e^t - t + 1)\vec{j} - t\vec{k}$.

(10pts) 5. (a) Find the equation of the line of intersection of the planes $x - y + z = 2$ and $2x + y + 3z = 1$.

Solution: The normals of the two planes are $\vec{n}_1 = \langle 1, -1, 1\rangle$ and $\vec{n}_2 = \langle 2, 1, 3\rangle$. The line of intersection must be perpendicular on both $\vec{n}_1$ and $\vec{n}_2$. Therefore we can chose the direction of the line to be the cross product of the normals:

$$\vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{z} \\ 1 & -1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -4\vec{i} - \vec{j} + 3\vec{k}.$$

We still need to find a point on the line. For this we start by setting of the coordinates to 0. For this example, it seems that if we make $z = 0$ we obtain easier equations. So let's set $z = 0$ and plug in into the equations of the planes. We obtain the following system of equation:

$$\begin{cases} x - y &= 2 \\ 2x + y &= 1. \end{cases}$$

Solving the system of equations we obtain $x = 1$ and $y = -1$. Thus a point of the line is $P(1, -1, 0)$. Therefore, the vector equation of the line is

$$\vec{r}(t) = \langle 1 - 4t, -1 - t, 3t\rangle.$$

(10pts) (b) Find the equation of the plane that passes through the point $P(2, 1, -1)$ and contains the line from part (a).

Solution: We know a point on the plane and we need to find the normal to the plane. We know that the line we found in part a) is contained in the plane. Thus the direction vector $\langle -4, -1, 3\rangle$ is parallel to the plane. To find a second vector parallel to the plane, we find first another point in the plane. We know from the first part that $Q(1, -1, 0)$ lies on the line and, hence, in the plane. Then $\vec{PQ} = \langle -1, -2, 1\rangle$ is the vector we want. Then the normal is the cross-product of $\vec{v}$ and $\vec{PQ}$. Thus $\vec{n} = 5\vec{i} + \vec{j} + 7\vec{k}$. Then an equation of the plane is

$$5(x - 2) + (y - 1) + 7(z + 1) = 0.$$