1. Let $f(x, y, z) = 2xyz + 2x + z$.

(15pts) (a) Find the directional derivative at $(1, 0, 1)$ in the direction from $(1, 0, 1)$ to the origin.

Solution: The gradient is the vector

$$\nabla f(1, 0, 1) = \langle f_x(1, 0, 1), f_y(1, 0, 1), f_z(1, 0, 1)\rangle.$$

We have that $f_x = 2yz + 2$, $f_y = 2xz$ and $f_z = 2xy + 1$. Thus

$$\nabla f(1, 0, 1) = \langle 2, 2, 1\rangle.$$

The direction is given by the vector starting at $(1, 0, 1)$ and ending at $(0, 0, 0)$. Thus the direction is given by the vector $v = \langle -1, 0, -1\rangle$. Recall that we actually need to find the corresponding unit vector. The length of v is $|v| = \sqrt{1 + 1} = \sqrt{2}$. Thus the unit vector is $u = \langle -\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}\rangle$. Finally, the directional derivative is

$$D_u f(1, 0, 1) = \nabla f(1, 0, 1) \cdot u = -2\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2} = -\frac{3\sqrt{2}}{2}.$$

(10pts) (b) Find a unit vector in the direction in which $f(x, y, z)$ increases most rapidly at the point $(1, 0, 1)$. What is the maximum rate of change at this point?

Solution: The direction of the maximum rate of change is given by the gradient we found in part a)

$$\nabla f(1, 0, 1) = \langle 2, 2, 1\rangle.$$

The maximum rate of change is given by the length of the gradient, $|\nabla f(1, 0, 1)| = 3$.

(25pts) 2. Find an equation of the tangent plane to the ellipsoid

$$\frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{16} = 6$$

at the point $(3, 4, -4)$.

Solution: The equation of the tangent plane to a level surface $F(x, y, z) = k$ is

$$F_x(x_0, y_0, y_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0.$$

In this problem

$$F(x, y, z) = \frac{x^2}{9} + \frac{y^2}{4} + \frac{z^2}{16}.$$

Then $F_x = 2x/9$, $F_y = y/2$, and $F_z = z/8$. So $F_x(3, 4, -4) = 2/3$, $F_y(3, 4 - 4) = 2$, and $F_z(3, 4, -4) = -1/2$. The equation of the tangent plane is

$$\frac{2}{3}(x - 3) + 2(y - 4) - \frac{1}{2}(z + 4) = 0,$$

(25pts) 3. Find all local maximum, minimum and saddle points of the function

$$f(x,y) = x^4 + y^4 - 4xy + 2.$$

Recall that if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$, then (x_0, y_0) is a local minimum point, if $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$, then (x_0, y_0) is a local maximum point and if $D(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point. Recall also that $D(x_0, y_0) = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - (f_{xy}(x_0, y_0))^2$.

Solution: We need to find first the critical points of f. The partial derivatives of f are $f_x(x,y) = 4x^3 - 4y$ and $f_y(x,y)4y^3 - 4x$. To find the critical points of f we need to solve the system of equations

$$\begin{cases} 4x^3 - 4y &= 0 \\ 4y^3 - 4x &= 0. \end{cases}$$

From the first equation we solve for y: $y = x^3$. We plug in into the second equation: $x^9 - x = 0$. Thus $x = 0$ or $x^8 = 1$. Therefore, the solutions are $x = 0, -1, 1$. Since $y = x^3$, the critical points of f are $(0,0)$, $(-1,-1)$, and $(1,1)$.

To determine whether these points are local maximum, minimum, or saddle points, we use the second derivative test. The second order derivatives of f are: $f_{xx}(x,y) = 12x^2$, $f_{xy}(x,y) = -4$, and $f_{yy}(x,y) = 12y^2$. We have that: if $(x,y) = (0,0)$, the determinant $D = -16 < 0$; thus $(0,0)$ is a saddle point. If $(x,y) = (-1,-1)$, then $D = 144 - 16 = 128 > 0$ and $f_{xx}(-1,-1) = 12 > 0$; thus $(-1,-1)$ is a local minimum point. Finally, if $(x,y) = (1,1)$, then $D = 144 - 16 = 128 > 0$ and $f_{xx}(1,1) = 12 > 0$; thus $(1,1)$ is also a local minimum point.

(25pts) 4. Suppose that $z = f(x, y)$ and $x = uv$ and $y = u + 3v$. Assume that, when $u = 2$ and $v = 1$, then

$$\frac{\partial z}{\partial u} = -2 \quad \text{and} \quad \frac{\partial z}{\partial v} = -1.$$

Find

$$\frac{\partial z}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y}$$

at $x = 2$ and $y = 5$.

Solution: Since f is a function of x and y, and these are functions of u and v, the chain rule for the partial derivatives of z with respect to u and v is

$$\begin{aligned} \frac{\partial z}{\partial u} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial v} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v} \end{aligned}$$

Since $\frac{\partial x}{\partial u} = v$ and $\frac{\partial x}{\partial v} = u$, for $u = 2$ and $v = 1$ we have that $\frac{\partial x}{\partial u}(2, 1) = 1$ and $\frac{\partial x}{\partial v}(2, 1) = 2$. Also, $\frac{\partial y}{\partial u} = 1$ and $\frac{\partial y}{\partial v} = 3$. So we obtain the following system:

$$\begin{aligned} -2 &= \frac{\partial z}{\partial x} \cdot 1 + \frac{\partial z}{\partial y} \cdot 1 \\ -1 &= \frac{\partial z}{\partial x} \cdot 2 + \frac{\partial z}{\partial y} \cdot 3. \end{aligned}$$

Multiplying the first equation by -2 and adding to the second we obtain that $\frac{\partial z}{\partial y} = 3$. Substituting this value in the first equation we see that $\frac{\partial z}{\partial x} = -5$.