LIST OF FORMULAS FOR MATH 113, FALL 2012, PART 2

- The **level curves** of a function f of two variables are the curves with equations $f(x, y) = k$, where k is constant.
- The **limit** of $f(x, y)$ as (x, y) approaches (a, b) is L and we write

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

  if for every number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that if $(x, y) \in D$ and $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x, y) - L| < \varepsilon$. We also write $f(x, y) \to L$.
- If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path $C_1$and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.
- A function f of two variables is called **continuous at** (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$.
- The partial derivatives are defined via

$$\begin{aligned} f_x(a, b) &= \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} \\ f_y(a, b) &= \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h} \end{aligned}$$

- To find f_x regard y as a constant and differentiate $f(x, y)$ with respect to x.
- To find f_y regard x as a constant and differentiate $f(x, y)$ with respect to y.
- An equation of the tangent plane to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

- The linear function whose graph is this tangent plane

$$L(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

  is called the **linearization** of f at (a, b)
- The approximation

$$f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

  is called the **linear approximation** or the **tangent plane approximation** of f at (a, b).
- If $z = f(x, y)$ and x changes from (a, b) to $(a + \Delta x, b + \Delta y)$, then the **increment** of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b)$$

- If $z = f(x, y)$, then f is **differentiable** at (a, b) if Δz can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

- For a differentiable function $z = f(x,y)$ we define the **differential** $\mathrm{d}z$, also called the **total differential**, is defined by
$$\mathrm{d}z = f_x(x,y)\mathrm{d}x + f_y(x,y)\mathrm{d}y = \frac{\partial z}{\partial x}\mathrm{d}x + \frac{\partial z}{\partial y}\mathrm{d}y,$$
where the **differentials** $\mathrm{d}x$ and $\mathrm{d}y$ are independent variables.
- If $\mathrm{d}x = \Delta x = x - a$ and $\mathrm{d}y = \Delta y = y - b$ the the differential of z is
$$\mathrm{d}z = f_x(a,b)(x-a) + f_y(a,b)(y-b)$$
- **Chain Rule, case 1:** Suppose that $z = f(x,y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and
$$\frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\partial f}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}t} + \frac{\partial f}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}t}.$$
- **Chain Rule, case 2:** Suppose t hat $z = f(x,y)$ is a differentiable function of x and y, where $x = g(s,t)$ and $y = h(s,t)$ are differentiable functions of s and t. Then
$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}\\ \frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\end{aligned}$$
- The **directional derivative** of f at (x_0, y_0) in the direction of a **unit** vector $\mathbf{u} = \langle a, b\rangle$ is
$$D_{\mathbf{u}}f(x_0,y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0,y_0)}{h}$$
if this limit exists.
- If f is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and
$$D_{\mathbf{u}}f(x,y) = f_x(x,y)a + f_y(x,y)b.$$
- If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then we can write $\mathbf{u} = \langle\cos\theta, \sin\theta\rangle$ and
$$D_{\mathbf{u}}f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta.$$
- If f is a function of two variables x and y, then the **gradient** of f is the vector function ∇f defined by
$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}.$$
- **The tangent plane to the level surface** $F(x,y,z) = k$ **at** $P(x_0,y_0,z_0)$ is the plane passes through P and has normal vector $\nabla F(x_0,y_0,z_0)$.
- The **normal line** to S at P is the line passing through P and perpendicular to the tangent plane.
- A function f has a **local maximum** at (a,b) if
$$f(x,y) \le f(a,b)$$
when (x,y) is near (a,b). The number $f(a,b)$ is called a **local maximum value.**

- A function f has a **local minimum** at (a, b) if
$$f(x, y) \geq f(a, b)$$
when (x, y) is near (a, b) and $f(a, b)$ is called a **local minimum value**.
- If
$$f(x, y) \leq f(a, b)$$
(or $f(x, y) \geq f(a, b)$) for all points (x, y) in the domain of f, then f has an **absolute maximum** (or **absolute minimum**) at (a, b).
- A point (a, b) is called a **critical point** (or **stationary point**) of f if $f_x(a, b) = 0$ and $f_y(a, b) = 0$.
- **Second Derivative Test:** Suppose the second partial derivatives of f are continuous on a disk with center (a, b), and suppose that $f_x(a, b) = 0$ and $f_y(a, b) = 0$. Let
$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx} f_{yy} - (f_{xy})^2.$$
  (1) If $D > 0$ and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
  (2) If $D > 0$ and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
  (3) If $D < 0$, then $f(a, b)$ is not a local maximum or minimum. In this case the point (a, b) is called a **saddle point** of f.