1. This problem asks you to compute the same limit using two different ways:

(a) Using the **Algebraic Limit Theorem**, prove that the limit of \(\frac{5n + 1}{4n + 5}\) is as expected.

**Solution:** We can simplify the sequence as follows:

\[
\frac{5n + 1}{4n + 5} = \frac{n(5 + 1/n)}{n(4 + 5/n)} = \frac{5 + 1/n}{4 + 5/n}.
\]

Using ALT, the sequence in the numerator and the sequence in the denominator are convergent, and \(\lim 5 + 1/n = 5\) and \(\lim 4 + 5/n = 4\). ALT implies that

\[
\lim \frac{5n + 1}{4n + 5} = \lim \frac{5 + 1/n}{4 + 5/n} = \frac{5}{4}.
\]

(b) Using the **definition**, prove that the limit of \(\frac{5n + 1}{4n + 5}\) is as expected.

**Solution:** Let \(\varepsilon > 0\). We need to find \(N \in \mathbb{N}\) such that if \(n \geq N\) then

\[
\left| \frac{5n + 1}{4n + 5} - \frac{5}{4} \right| < \varepsilon.
\]

The inequality above is equivalent to

\[
\frac{20n + 4 - 20n - 25}{4(4n + 5)} < \varepsilon \iff \frac{21}{4(4n + 5)} < \varepsilon \iff \frac{21}{4\varepsilon} < 4n + 5 \iff \frac{1}{4} \left( \frac{21}{4\varepsilon} - 5 \right) < n.
\]

Let \(N\) to be the smallest positive integer greater or equal than \(\frac{1}{4} \left( \frac{21}{4\varepsilon} - 5 \right)\). Then, if \(n \geq N\) we have that

\[
\left| \frac{5n + 1}{4n + 5} - \frac{5}{4} \right| < \varepsilon.
\]

Therefore, according to the definition, we have that \(\lim(5n + 1)/(4n + 5) = 5/4\).
2. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorems:

(a) A convergent sequence \((a_n)\) such that \(|a_n - a_{n+1}| > 0.01\) for infinitely many \(n\).

**Solution:** This request is impossible. According to the Cauchy criterion, a sequence is convergent iff it is Cauchy. If a sequence is Cauchy, then, for \(\epsilon = 0.01\), there is \(N \in \mathbb{N}\) such that if \(m > n \geq N\) then \(|a_n - a_m| < 0.01\). Letting \(m = n + 1\) in this inequality we obtain a contradiction.

(b) A Cauchy sequence with an unbounded subsequence.

**Solution:** This request is impossible: a Cauchy sequence is bounded; therefore, any subsequence of the sequence is bounded.

(c) Two sequences \((a_n)\) and \((b_n)\) where \((a_n/b_n)\) and \((b_n)\) converge, but \((a_n)\) does not.

**Solution:** This request is impossible. According to the ALT, if two sequences converge, then so is their product. Since \(a_n/b_n \cdot b_n = a_n\), it follows that \((a_n)\) converges, which is a contradiction.

(d) A set \(A\) for which there is no sequence in \(A\) with limit sup \(A\) that is eventually constant.

**Solution:** Any set that does not contain the sup would do it. For example \(A = (0,1)\).
3. Show that $\sqrt{\sqrt{2} + 2}$ is irrational.

**Solution:** Assume, by contradiction, that $\sqrt{\sqrt{2} + 2}$ is irrational. Then there is a rational $r$ such that $\sqrt{\sqrt{2} + 2} = r$. If follows that $\sqrt{2} + 2 = r^2$ and $\sqrt{2} = r^2 - 2$. Since $\mathbb{Q}$ is a field, it follows that $r^2 - 2$ is a rational number. Hence $\sqrt{2} \in \mathbb{Q}$, which is a contradiction.
4. Let $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$ and let $B = \{b \in \mathbb{Q} \mid b^2 > 2\}$. Show that $\sup A = \inf B$.

**Solution:** Let $a \in A$ and $b \in B$. Then, since $a^2 < 2 < b^2$ it follows that $a < b$. Since $b$ was arbitrary, we have that $a$ is a lower bound for $B$. Therefore $a \leq \inf B$ for all $a \in A$. It follows that $\inf B$ is an upper bound for $A$. Therefore, $\sup A \leq \inf B$. Assume, by contradiction, that $\sup A < \inf B$. Then there is a rational number $r$ such that $\sup A < r < \inf B$. Since $r \notin A$, it follows that $r^2 \geq 2$. Since $r \notin B$ (and $r > 0$) it follows that $r^2 \leq 2$. Therefore $r^2 = 2$, which is a contradiction with the fact that $\sqrt{2}$ is irrational. Therefore $\sup A = \inf B$. 
5. Let us say that a sequence \((c_n)_{n=1}^{\infty}\) of real numbers "cervonges to \(c\)" (where \(c \in \mathbb{R}\)) if and only if there is an \(N \in \mathbb{N}\) such that, for all \(n > N\) and all \(\varepsilon > 0\), \(|c_n - c| < \varepsilon\).

(a) If a sequence \((c_n)\) cervonges to \(c\), does \((c_n)\) converge to \(c\)? Explain, and if not, give an example.

\textbf{Solution:} The \textbf{main} difference between cervongent and convergent sequences is that, for a cervongent sequence, \(N\) \textit{does not} depend on \(\varepsilon\), while for convergent sequences it does. That is, for a cervongent sequence, the same \(N\) works for all \(\varepsilon > 0\). This can happen if and only if \(c_n = c\) is constant for \(n \geq N\). Thus a cervongent sequence is convergent.

(b) If a sequence \((c_n)\) converges to \(c\), does \((c_n)\) cervonge to \(c\)? Explain, and if not, give an example.

\textbf{Solution:} In general, a convergent sequence is not cervongent. Any convergent sequence that is not eventually constant would work as an example. E.g.: \(c_n = 1/n\).