1. This problem asks you to compute the same limit using two different ways:

(10pts) (a) Using the *Algebraic Limit Theorem*, prove that the limit of $(5n+1)/(4n+5)$ is as expected.

Solution: We can simplify the sequence as follows:

$$\frac{5n+1}{4n+5} = \frac{n(5+1/n)}{n(4+5/n)} = \frac{5+1/n}{4+5/n}.$$

Using ALT, the sequence in the numerator and the sequence in the denominator are convergent, and $\lim 5+1/n = 5$ and $\lim 4+5/n = 4$. ALT implies that

$$\lim \frac{5n+1}{4n+5} = \lim \frac{5+1/n}{4+5/n} = \frac{\lim 5+1/n}{4+5/n} = \frac{5}{4}.$$

(10pts) (b) Using the *definition*, prove that the limit of $(5n+1)/(4n+5)$ is as expected.

Solution: Let $\varepsilon > 0$. We need to find $N \in \mathbb{N}$ such that if $n \geq N$ then

$$\left|\frac{5n+1}{4n+5} - \frac{5}{4}\right| < \varepsilon.$$

The inequality above is equivalent to

$$\begin{aligned}
&\left|\frac{20n+4-20n-25}{4(4n+5)}\right| < \varepsilon \\
\Leftrightarrow &\frac{21}{4(4n+5)} < \varepsilon \\
\Leftrightarrow &\frac{21}{4\varepsilon} < 4n+5 \\
\Leftrightarrow &\frac{1}{4}\left(\frac{21}{4\varepsilon} - 5\right) < n.
\end{aligned}$$

Let N to be the smallest positive integer greater or equal than $\frac{1}{4}\left(\frac{21}{4\varepsilon} - 5\right)$. Then, if $n \geq N$ we have that

$$\left|\frac{5n+1}{4n+5} - \frac{5}{4}\right| < \varepsilon.$$

Therefore, according to the definition, we have that $\lim(5n+1)/(4n+5) = 5/4$.

2. Give an example of each of the following, or state that such a request is impossible by referencing the proper theorems:

(5pts) (a) A convergent sequence (a_n) such that $|a_n - a_{n+1}| > 0.01$ for infinitely many n.

> **Solution:** This request is impossible. According to the Cauchy criterion, a sequence is convergent iff it is Cauchy. If a sequence is Cauchy, then, for $\varepsilon = 0.01$, there is $N \in \mathbb{N}$ such that if $m > n \geq N$ then $|a_n - a_m| < 0.01$. Letting $m = n + 1$ in this inequality we obtain a contradiction.

(5pts) (b) A Cauchy sequence with an unbounded subsequence.

> **Solution:** This request is impossible: a Cauchy sequence is bounded; therefore, any subsequence of the sequence is bounded.

(5pts) (c) Two sequences (a_n) and (b_n) where (a_n/b_n) and (b_n) converge, but (a_n) does not.

> **Solution:** This request is impossible. According to the ALT, if two sequences converge, then so is their product. Since $a_n/b_n \cdot b_n = a_n$, if follows that (a_n) converges, which is a contradiction.

(5pts) (d) A set A for which there is no sequence in A with limit $\sup A$ that is eventually constant.

> **Solution:** Any set that does not contain the sup would do it. For example $A = (0, 1)$.

(15pts) 3. Show that $\sqrt{\sqrt{2}+2}$ is irrational.

Solution: Assume, by contradiction, that $\sqrt{\sqrt{2}+2}$ is irrational. Then there is a rational r such that $\sqrt{\sqrt{2}+2} = r$. If follows that $\sqrt{2}+2 = r^2$ and $\sqrt{2} = r^2 - 2$. Since $\mathbb{Q}$ is a field, it follows that $r^2 - 2$ is a rational number. Hence $\sqrt{2} \in \mathbb{Q}$, which is a contradiction.

(25pts) 4. Let $A = \{a \in \mathbb{Q} \mid a^2 < 2\}$ and let $B = \{b \in \mathbb{Q} \mid b^2 > 2\}$. Show that $\sup A = \inf B$.

Solution: Let $a \in A$ and $b \in B$. Then, since $a^2 < 2 < b^2$ it follows that $a < b$.Since b was arbitrary, we have that a is a lower bound for B. Therefore $a \leq \inf B$ for all $a \in A$. It follows that $\inf B$ is an upper bound for A. Therefore, $\sup A \leq \inf B$. Assume, by contradiction, that $\sup A < \inf B$. Then there is a rational number r such that $\sup A < r < \inf B$. Since $r \notin A$, it follows that $r^2 \geq 2$. Since $r \notin B$ (and $r > 0$) it follows that $r^2 \leq 2$. Therefore $r^2 = 2$, which is a contradiction with the fact that $\sqrt{2}$ is irrational. Therefore $\sup A = \inf B$.

5. Let us say that a sequence $(c_n)_{n=1}^{\infty}$ of real numbers *"cervonges* to c" (where $c \in \mathbb{R}$) if and only if there is an $N \in \mathbb{N}$ such that, for all $n > N$ and all $\varepsilon > 0$, $|c_n - c| < \varepsilon$.

(10pts) (a) If a sequence (c_n) cervonges to c, does (c_n) converge to c? Explain, and if not, give an example.

> **Solution:** The **main** difference between cervongent and convergent sequences is that, for a cervongent sequence, N *does not* depend on ε, while for convergent sequences it does. That is, for a cervongent sequence, the same N works for all $\varepsilon > 0$. This can happen if and only if $c_n = c$ is constant for $n \geq N$. Thus a cervongent sequence is convergent.

(10pts) (b) If a sequence (c_n) converges to c, does (c_n) cervonge to c? Explain, and if not, give an example.

> **Solution:** In general, a convergent sequence is not cervongent. Any convergent sequence that is not eventually constant would work as an example. E.g.: $c_n = 1/n$.