1. True or false? Give a short explanation.

(a) $\sum (-1)^{n+1}(1/n^p)$ converges for all $p > 0$.  
(b) The set of irrational numbers is a closed set.  
(c) If the sequence $({b}_n)$ converges to $b$, then the set $B = \{b, {b}_1, {b}_2, {b}_3, \ldots \}$ is a closed set.  
(d) The set $\{(-1)^n(1 - \frac{1}{n}) : n \in \mathbb{N}\}$ is an open set.

Solution:

(a) True, by the Alternating Series Test. (It converges absolutely only for $p > 1$.)

(b) False: For example, $(\sqrt{2}/n)$ is a sequence of irrationals converging to the rational 0.

(c) True: The only limit point of $B$ is $b$, which is in the set.

(d) False: It does not contain open intervals around any of its elements.
2. (a) (10pts) Determine whether the series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+3)}{n^3 - 5n + 6} \]
converges absolutely, converges conditionally, or diverges, and give a reason.

**Solution:** Intuitively, this series should behave like the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \). Therefore, we expect that it is absolutely convergent. We will use the comparison test. If \( n \geq 3 \), then
\[ \frac{n+3}{n^3 - 5n + 6} \leq \frac{2n}{n^3 - 5n} = \frac{2}{n^2 - 5} \leq \frac{5}{n^2}. \]
The last inequality is also valid only for \( n \geq 3 \). Since \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converges, it follows using the ALT that \( \sum_{n=1}^{\infty} \frac{n+3}{n^3 - 5n + 6} \) converges. Therefore, \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n+3)}{n^3 - 5n + 6} \) is absolutely convergent.

(10pts) (b) Prove that
\[ \sum_{n=2}^{\infty} \frac{1}{n \log^p(n)} \]
converges for all \( p > 1 \).

**Solution:** We can use the integral test to solve this problem. Let \( f(x) = \frac{1}{x \log^p(x)} \) (we assume that we have natural log). Then, using the \( u \)-substitution \( u = \log(x) \), we have that
\[ \int_{1}^{n} f(x) \, dx = \int_{1}^{\log(n)} \frac{1}{u^{p+1}} \, du = \begin{cases} \frac{(\log n)^{p+1}}{-p+1} & \text{if } p \neq 1 \\ \log(\log(n)) & \text{if } p = 1. \end{cases} \]
Therefore, the limit \( \lim_{n \to \infty} \int_{1}^{n} f(x) \, dx \) exists (and is finite) iff \( p > 1 \). Thus, by the integral test, the series \( \sum_{n=2}^{\infty} \frac{1}{n \log^p(n)} \) converges iff \( p > 1 \).
3. If \( A \) is bounded, prove that \( \overline{A} \) is bounded.

Solution: Let \( M > 0 \) such that \( |x| < M \) for all \( x \in A \). We need to find \( M' \) such that \( |x| < M' \) for all \( x \in \overline{A} \). Let \( x \in \overline{A} \). If \( x \in A \), then \( |x| < M \). If \( x \notin A \), then \( x \) is a limit point of \( A \). Therefore, there exists a sequence \( (x_n) \) in \( A \) such that \( \lim x_n = x \). Since \( |x_n| < M \), it follows from the Order Limit Theorem (Theorem 2.3.4 in the textbook) that \( |x| \leq M \). Therefore \( |x| < M + 1 \) for all \( x \in \overline{A} \), and \( \overline{A} \) is bounded.
4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that there are two numbers $c \leq d$ such that $f([a, b]) = [c, d]$ (the image of a closed interval is a closed interval).

**Solution:** We know from class that $[a, b]$ is a compact set. Since $f$ is continuous, there are two numbers $x_m$ and $x_M$ in $[a, b]$ such that $f(x_m)$ is the minimum value of $f$ on $[a, b]$ and $f(x_M)$ is the maximum value of $f$ on $[a, b]$. Let $x = f(x_m)$ and $d = f(x_M)$. Since $c \leq f(x) \leq d$ for all $x \in [a, b]$ it follows that $f([a, b]) \subset [c, d]$. We need to prove the converse inclusion as well. If $c = d$, then $f$ is constant and the result is trivially true. Assume now that $c < d$. Let $y \in [c, d]$. By the intermediate value theorem, there is a number $x$ between $x_m$ and $x_M$ such that $f(x) = y$. Then $x$ has to belong to $[a, b]$. Therefore $y \in f([a, b])$ and we have that $f([a, b]) \supset [c, d]$. Thus $f([a, b]) = [c, d]$. 
(20pts) 5. Suppose that \( f : \mathbb{R} \rightarrow \mathbb{R} \) is continuous and let \( L \in \mathbb{R} \). Prove that \( \{ x : f(x) < L \} \) is an open set.

**Solution:** Let \( c \in \{ x : f(x) < L \} \). We need to find a neighborhood of \( c \) completely contained in the set. Let \( \varepsilon = L - f(c) \). Since \( f \) is continuous, there is \( \delta > 0 \) such that if \( |x - c| < \delta \), then \( |f(x) - f(c)| < \varepsilon \). Therefore \( f(x) < \varepsilon + f(x) = L - f(x) + f(x) = L \) for all \( x \in V_\delta(c) \). Therefore \( V_\delta(c) \subset \{ x : f(x) < L \} \). So our set is open.