1. True or false? Give a short explanation.

(5pts) (a) $\sum(-1)^{n+1}(1/n^p)$ converges for all $p > 0$.

(5pts) (b) The set of irrational numbers is a closed set.

(5pts) (c) If the sequence (b_n) converges to b, then the set $B = \{b, b_1, b_2, b_3, \ldots\}$ is a closed set.

(5pts) (d) The set $\{(-1)^n(1 - \frac{1}{n}) : n \in \mathbb{N}\}$ is an open set.

Solution:

(a) True, by the Alternating Series Test. (It converges absolutely only for $p > 1$.)

(b) False: For example, $(\sqrt{2}/n)$ is a sequence of irrationals converging to the rational 0.

(c) True: The only limit point of B is b, which is in the set.

(d) False: It does not contain open intervals around any of its elements.

(10pts) 2. (a) Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n^3-5n+6}$$

converges absolutely, converges conditionally, or diverges, and give a reason.

Solution: Intuitively, this series should behave like the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$. Therefore, we expect that it is absolutely convergent. We will use the comparison test. If $n \geq 3$, then $n+3 \leq 2n$. Also, $n^3-5n+6 \geq n^3-5n$. Therefore, for $n \geq 3$, we have that

$$\frac{n+3}{n^3-5n+6} \leq \frac{2n}{n^3-5n} = \frac{2}{n^2-5} \leq \frac{5}{n^2}.$$

The last inequality is also valid only for $n \geq 3$. Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges, it follows using the ALT that $\sum_{n=1}^{\infty}\frac{5}{n^2}$ converges. The comparison test implies that $\sum_{n=1}^{\infty}\frac{n+3}{n^3-5n+6}$ converges. Therefore $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n+3}{n^3-5n+6}$ is absolutely convergent.

(10pts) (b) Prove that

$$\sum_{n=2}^{\infty}\frac{1}{n\log^p(n)}$$

converges for all $p > 1$.

Solution: We can use the integral test to solve this problem. Let $f(x) = \frac{1}{x\log^p(x)}$ (we assume that we have natural log).Then, using the u-substitution $u = \log(x)$, we have that

$$\int_1^n f(x)\,\mathrm{d}\,x = \int_1^n \frac{1}{x\log^p(x)}\,\mathrm{d}\,x = \int_0^{\log(n)}\frac{1}{u^p}du = \begin{cases}\frac{(\log n)^{-p+1}}{-p+1} & \text{if } p \neq 1\\ \log(\log(n)) & \text{if } p = 1.\end{cases}$$

Therefore, the limit $\lim_{n\to\infty}\int_1^n f(x)\,\mathrm{d}\,x$ exists (and is finite) iff $p > 1$. Thus, by the integral test, the series $\sum_{n=2}^{\infty}\frac{1}{n\log^p(n)}$ converges iff $p > 1$.

(20pts) 3. If A is bounded, prove that $\overline{A}$ is bounded.

Solution: Let $M > 0$ such that $|x| < M$ for all $x \in A$. We need to find M' such that $|x| < M'$ for all $x \in \overline{A}$. Let $x \in \overline{A}$. If $x \in A$, then $|x| < M$. If $x \notin A$, then x is a limit point of A. Therefore, there exists a sequence (x_n) in A such that $\lim x_n = x$. Since $|x_n| < M$, it follows from the Order Limit Theorem (Theorem 2.3.4 in the textbook) that $|x| \leq M$. Therefore $|x| < M + 1$ for all $x \in \overline{A}$, and $\overline{A}$ is bounded.

(20pts) 4. Let $f : [a, b] \to \mathbb{R}$ be continuous. Show that there are two numbers $c \le d$ such that $f([a, b]) = [c, d]$ (the image of a closed interval is a closed interval).

Solution: We know from class that $[a, b]$ is a compact set. Since f is continuous, there are two numbers x_m and x_M in $[a, b]$ such that $f(x_m)$ is the minimum value of f on $[a, b]$ and $f(x_M)$ is the maximum value of f on $[a, b]$. Let $x = f(x_m)$ and $d = f(x_M)$. Since $c \le f(x) \le d$ for all $x \in [a, b]$ it follows that $f([a, b]) \subset [c, d]$. We need to prove the converse inclusion as well. If $c = d$, then f is constant and the result is trivially true. Assume now that $c < d$. Let $y \in [c, d]$. By the intermediate value theorem, there is a number x between x_m and x_M such that $f(x) = y$. Then x has to belong to $[a, b]$. Therefore $y \in f([a, b])$ and we have that $f([a, b]) \supset [c, d]$. Thus $f([a, b]) = [c, d]$.

(20pts) 5. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous and let $L \in \mathbb{R}$. Prove that $\{x : f(x) < L\}$ is an open set.

Solution: Let $c \in \{x : f(x) < L\}$. We need to find a neighborhood of c completely contained in the set. Let $\varepsilon = L - f(c)$. Since f is continuous, there is $\delta > 0$ such that if $|x - c| < \delta$, then $|f(x) - f(c)| < \varepsilon$. Therefore $f(x) < \varepsilon + f(x) = L - f(x) + f(x) = L$ for all $x \in V_\delta(c)$. Therefore $V_\delta(c) \subset \{x : f(x) < L\}$. So our set is open.