HOMEWORK # 2

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Problem 1 (Exercise 1.4.2 on page 27)**.** Recall that $\mathbb{I}$ stands for the set of irrational numbers.

(a) Show that if $a, b \in \mathbb{Q}$, then ab and $a+b$ are elements of $\mathbb{Q}$ as well.

(b) Show that if $a \in \mathbb{Q}$ and $t \in \mathbb{I}$, then $a+t \in I$ and $at \in \mathbb{I}$ as long as $a \neq 0$.

(c) Part (a) can be summarized by saying that $\mathbb{Q}$ is closed under addition and multiplication. Is $\mathbb{I}$ closed under addition and multiplication? Given two irrational numbers s and t, what can we say about $s+t$ and st?

Problem 2 (Exercise 1.4.4 on page 27)**.** Use the Archimedean Property of $\mathbb{R}$ to rigorously prove that $\inf\{1/n : n \in \mathbb{N}\} = 0$.

Problem 3 (Exercise 2.2.1 on page 43)**.** Verify, using the definition of convergence of a sequence, that the following sequences converge to the proposed limit.

(a) $\lim \frac{1}{(6n^2+1)} = 0$.

(b) $\lim \frac{3n+1}{2n+5} = \frac{3}{2}$.

(c) $\lim \frac{2}{\sqrt{n+3}} = 0$.

Problem 4 (Exercise 2.2.2 on page 43)**.** What happens if we reverse the order of the quantifiers in Definition 2.2.3?

Definition: A sequence (x_n) *verconges* to x if *there exists* an $\varepsilon > 0$ such that *for all* $N \in \mathbb{N}$ it is true that $n \geq N$ implies $|x_n - x| < \varepsilon$.

Give an example of a vercongent sequence. Can you give an example of a vercongent sequence that is divergent? What exactly is being described in this strange definition?

Problem 5 (Exercise 2.2.6 in page 43)**.** Suppose that for a particular $\varepsilon > 0$ we have found a suitable value of N that "works" for a given sequence in the sense of Definition 2.2.3.

(a) Then, any larger/smaller *pick one)* N will also work for the same $\varepsilon > 0$.

(b) Then, this same N will also work for any larger/smaller value of ε.

Problem 6 (Exercise 2.2.7 on page 45)**.** Informally speaking, the sequence $\sqrt{n}$ " converges to infinity."

(a) Imitate the logical structure of Definition 2.2.3 to create a rigorous definition for the mathematical statement $\lim x_n = \infty$. Use this definition to prove $\lim \sqrt{n} = \infty$.

(b) What does your definition in (a) say about the particular sequence $\{1, 0, 2, 0, 3, 0, 4, 0, 5, 0 \dots\}$?