HOMEWORK #4, REAL ANALYSIS I, FALL 2012

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Problem 1 (Exercise 2.4.2 on page 54)**.**

(a) Prove that the sequence defined by $x_1 = 3$ and

$$x_{n+1} = \frac{1}{4 - x_n}$$

converges.

(b) Now that we know $\lim x_n$ exists, explain why $\lim x_{n+1}$ must also exist and equal the same value.

(c) Take the limit of each side of the recursive equation in part (a) of this exercise to explicitly compute $\lim x_n$.

Problem 2 (Exercise 2.4.4 on page 54)**.** Show that

$$\sqrt{2},\ \sqrt{2\sqrt{2}},\ \sqrt{2\sqrt{2\sqrt{2}}},\ \ldots$$

converges and find the limit.

Problem 3 (Exercise 2.4.5 on page 54, **(Calculating Square Roots).**)**.** Let $x_1 = 2$, and define

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right) .$$

(a) Show that x_n^2 is always greater than 2, and then use this to prove that $x_n - x_{n+1} \geq 2$. Conclude that $\lim x_n = \sqrt{2}$.

(b) Modify the sequence (x_n) so that it converges to $\sqrt{c}$.

Problem 4 (Exercise 2.5.3 on page 58)**.** Give an example of each of the following, or argue that such a request is impossible

(a) A sequence that does not contain 0 or 1 as a term but contains subsequences converging to each of these values.

(b) A monotone sequence that diverges but has a convergent subsequence.

(c) A sequence that contains subsequences converging to every point in the infinite set $\{1, 1/2, 1/3, 1/4, 1/5, \ldots\}$.

(d) An unbounded sequence with a convergent subsequence.

Date: **Due Monday, 10/01/2012; Note that it is due on Monday instead of Friday!**

(e) A sequence that has a subsequence that is bounded but contains no subsequence that converges.

Problem 5 (Exercise 2.6.1 on page 61)**.** Give an example of each of the following, or argue that such a request is impossible.

(a) A Cauchy sequence that is not monotone.
(b) A monotone sequence that is not Cauchy.
(c) A Cauchy sequence with a divergent subsequence.
(d) An unbounded sequence containing a subsequence that is Cauchy.

Problem 6 (Exercise 2.6.3 on page 61)**.** (a) Explain how the following pseudo-Cauchy property differs from the proper definition of a Cauchy sequence: A sequence (a_n) is *pseudo-Cauchy* if, for all $\varepsilon > 0$, there exists an N such that if $n \geq N$, then $|s_{n+1} - s_n| < \varepsilon$.
(b) If possible, give an example of a divergent sequence (s_n) that is pseudo-Cauchy.