HOMEWORK #5, REAL ANALYSIS I, FALL 2012

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Problem 1 (Exercise 2.7.1 on page 67; **This problem is worth double the points (12points)**)**.** Proving the Alternating Series Test (Theorem 2.7.7) amounts to showing that the sequence of partial sums

$$s_n = a_1 - a_2 + a_3 - \cdots \pm a_n$$

converges. Different characterization of completeness lead to different proofs. **Please choose one of the following methods to prove the Alternating Series Test**:

(a) Prove the Alternating Series Test by showing that (s_n) is a Cauchy sequence.
(b) Prove the Alternating Series Test using the Nested Interval Property (Theorem 1.4.1)
(c) Consider the subsequences (s_{2n}) and (s_{2n+1}), and show how the Monotone Convergence Theorem leads to a third proof for the Alternating Series Test.

Problem 2 (Exercise 2.7.4 on page 68)**.** Give an example to show that it is possible for both $\sum x_n$ and $\sum y_n$ to diverge but for $\sum x_n y_n$ to converge.

Problem 3 (Exercise 2.7.5 on page 68)**.** (a) Show that if $\sum a_n$ converges absolutely, then $\sum a_n^2$ also converges absolutely. Does this proposition hold without absolute convergence?
(b) If $\sum a_n$ converges and $a_n \geq 0$, can we conclude anything about $\sqrt{a_n}$?

Problem 4 (Exercise 2.7.9 on page 68 (**Ratio Test**))**.** Given a series $\sum_{n=1}^{\infty} a_n$ with $a_n \neq 0$, the Ratio Test states that if (a_n) satisfies

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = r < 1 ,$$

then the series converges absolutely.

(a) Let r' satisfy $r < r' < 1$. (Why must such an r' exist?) Explain why there exists an N such that $n \geq N$ implies $|a_{n+1}| \leq |a_n| r'$.
(b) Why does $|a_N| \sum (r')^n$ necessarily converge?

Date: **Due Friday, 10/19/2012.**

(c) Now, show that $\sum |a_n|$ converges.

Problem 5 (This problem is not from the textbook)**.** Test each of the following series for absolute convergence, conditional convergence, or divergence:

(a)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n^2 - 5n + 1}$$

(b)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(n-5)}{n^3 - 7n - 9}$$

(c)
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+1}$$