Homework #6, Real Analysis I, Fall 2012

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Problem 1 (Exercise 3.2.2 on page 83). Let

\[ B = \left\{ \frac{(-1)^n n}{n+1} : n = 1, 2, 3, \ldots \right\} . \]

(a) Find the limit points of \( B \).
(b) Is \( B \) a closed set?
(c) Is \( B \) an open set?
(d) Find \( B \).

Problem 2 (Exercise 3.2.3 on page 83). Decide whether the following sets are open, closed, or neither. If a set is not open, find a point in the set for which there is no \( \varepsilon \)-neighborhood contained in the set. If a set is not closed, find a limit point that is not contained in the set.

(a) \( \mathbb{Q} \).
(b) \( \mathbb{N} \).
(c) \( \{ x \in \mathbb{R} : x > 0 \} \).
(d) \( [0, 1] = \{ x \in \mathbb{R} : 0 < x \leq 1 \} \).
(e) \( \{ 1 + 1/4 + 1/9 + \cdots + 1/n^2 : n \in \mathbb{N} \} \).

Problem 3 (Exercise 3.2.7 on page 83). Let \( x \in O \), where \( O \) is an open set. If \( (x_n) \) is a sequence converging to \( x \), prove that all but a finite number of the terms of \( (x_n) \) must be contained in \( O \).

Problem 4 (Exercise 3.2.8 on page 83). Given \( A \subseteq \mathbb{R} \), let \( L \) be the set of all limit points of \( A \).

(a) Show that the set \( L \) is closed.
(b) Argue that if \( x \) is a limit point of \( A \cup L \), then \( x \) is a limit point of \( A \). Use this observation to furnish a proof for Theorem 3.2.12.

Problem 5. (a) If \( y \) is a limit point of \( A \cup B \), show that \( y \) is either a limit point of \( A \) or a limit point of \( B \) (or both).
(b) Prove that \( \overline{A \cup B} = \overline{A} \cup \overline{B} \).
(c) Does the result about closures in (b) extend to infinite unions of sets?

Date: Due Friday, 10/26/2012.
Problem 6. Decide whether the following statements are true or false. Provide counterexamples for those that are false, and supply proofs for those that are true.

(a) For any set $A \subseteq \mathbb{R}$, $\overline{A^c}$ is open.
(b) If a set $A$ has an isolated point, it cannot be an open set.
(c) A set $A$ is closed if and only if $\overline{A} = A$.
(d) If $A$ is a bounded set, then $s = \sup A$ is a limit point of $A$.
(e) Every finite set is closed.
(f) An open set that contains every rational number must necessarily be all of $\mathbb{R}$.