HOMEWORK #7, REAL ANALYSIS I, FALL 2012

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Problem 1 (Exercise 4.3.5 on page 113)**.** In Theorem 4.3.4, statement (iv) says that $f(x)/g(x)$ is continuous at c if both f and g are, provided that the quotient is defined. Show that if g is continuous at c and $g(c) \neq 0$, then there exists an open interval containing c on which $f(x)/g(x)$ is defined.

Problem 2 (Exercise 4.3.7 on page 113)**.** Assume that $h : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and let $K := \{x : h(x) = 0\}$. Show that K is a closed set.

Problem 3 (Exercise 4.4.4 on page 119)**.** Show that if f is continuous on $[a, b]$ with $f(x) > 0$ for all $a \leq x \leq b$, then $1/f$ is bounded on $[a, b]$.

Problem 4 (Exercise 4.4.8 on page 119)**.**

(a) Assume that $f : [0, \infty) \to \mathbb{R}$ is continuous at every point in its domain. Show that if there exists $b > 0$ such that f is uniformly continuous on the set $[b, \infty)$, then f is uniformly continuous on $[0, \infty)$.

(b) Prove that $f(x) = \sqrt{x}$ is uniformly continuous on $[0, \infty)$.

Problem 5 (Exercise 4.4.9 on page 120)**.** A function $f : A \to \mathbb{R}$ is called *Lipschitz* if there exists a bound $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x, y \in A$. Geometrically speaking, a function f is Lipschitz if there is a uniform bound on the magnitude of the slopes of lines drawn through any two points on the graph of f.

(a) Show that if $f : A \to \mathbb{R}$ is Lipschitz, then it is uniformly continuous on A.

(b) Is the converse statement true? Are all uniformly continuous functions necessarily Lipschitz?

Problem 6 (Exercise 4.5.3 on page 124)**.** Is there a continuous function on all of $\mathbb{R}$ with range $f(\mathbb{R})$ equal to $\mathbb{Q}$?

Date: **Due Friday, 11/9/2012.**