HOMEWORK #9, REAL ANALYSIS I, FALL 2012

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Problem 1 (Exercise 5.2.2 on page 136)**.**
(a) Use Definition 5.2.1 to product the proper formula for the derivative of $f(x) = 1/x$.
(b) Combine the result of (a) with the chain rule (Theorem 5.2.5) to supply a proof for part (iv) of Theorem 5.2.4 [the derivative rule for quotients].
(c) Supply a direct proof of Theorem 5.2.4 (iv) by algebraically manipulating the difference quotient for (f/g) in a style similar to the proof of Theorem 5.2.4 (iii).

Problem 2 (Exercise 4.2.4 on page 136)**.** Let $f_a(x) = \begin{cases} x^a & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$.
(a) For which values of a is f continuous at zero?
(b) For which values of a is f differentiable at zero? In this case, is the derivative function continuous?
(c) For which values of a is f twice-differentiable?

Problem 3 (Exercise 5.3.1 on page 143)**.** Recall from Exercise 4.4.9 that a function $f : A \to \mathbb{R}$ is "Lipschitz on A" if there exists an $M > 0$ such that

$$\left| \frac{f(x) - f(y)}{x - y} \right| \leq M$$

for all $x, y \in A$. Show that if f is differentiable on a closed interval $[a, b]$ and if f' is continuous on $[a, b]$, then f is Lipschitz on $[a, b]$.

Problem 4 (Exercise 5.3.5 on page 143)**.** A *fixed point* of a function f is a value x where $f(x) = x$. Show that if f is differentiable on an interval with $f'(x) \neq 1$, then f can have at most one fixed point.

Problem 5 (Exercise 5.3.7 on page 143)**.** Recall that a function $f : (a, b) \to \mathbb{R}$ is *increasing* on (a, b) if $f(x) \leq f(y)$ whenever $x \leq y$ in (a, b). Assume f is differentiable on (a, b). Show that f is increasing on (a, b) if and only if $f'(x) \geq 0$ for all $x \in (a, b)$.

Problem 6 (Exercise 6.2.1 on page 160)**.** Let

$$f_n(x) = \frac{nx}{1 + nx^2} \ .$$

Date: **Due Friday, 11/30/2012.**

(a) Find the pointwise limit of (f_n) for all $x \in (0, \infty)$.
(b) Is the convergence uniform on $(0, \infty)$?
(c) Is the convergence uniform on $(0, 1)$?
(d) Is the convergence uniform on $(1, \infty)$?