FINAL PROJECT TOPICS
MATH 399, SPRING 2011

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If you pick any of the following topics feel free to discuss with me if you need any further background that we did not discuss in class.

1. Iterations of Dynamical Systems and Chaos

For $\alpha \in [0,2]$ define the tent map

$$T_\alpha(x) = \begin{cases} \alpha x & 0 \leq x < \frac{1}{2} \\ \alpha(1-x) & \frac{1}{2} \leq x \leq 1 \end{cases}.$$

We discussed in class a few properties for $\alpha = 2$. Possible problems and questions for a project

- For what values of α are periodic points of period N for any $N \geq 1$?
- For what values of α are aperiodic orbits?
- Is the system is sensitive to the initial conditions? What is the error propagation?
- For what values of α does T_α satisfy the mixing property? (mixing property: for any two intervals I and J one can find initial values in I which, when iterated, will eventually lead to points in J. A dynamical system that is mixing, sensitive to the initial conditions, and has dense periodic orbits is called a *chaotic* dynamical system.
- Prove that mixing implies sensitive to the initial conditions.
- Use the computer to draw orbits of each type.

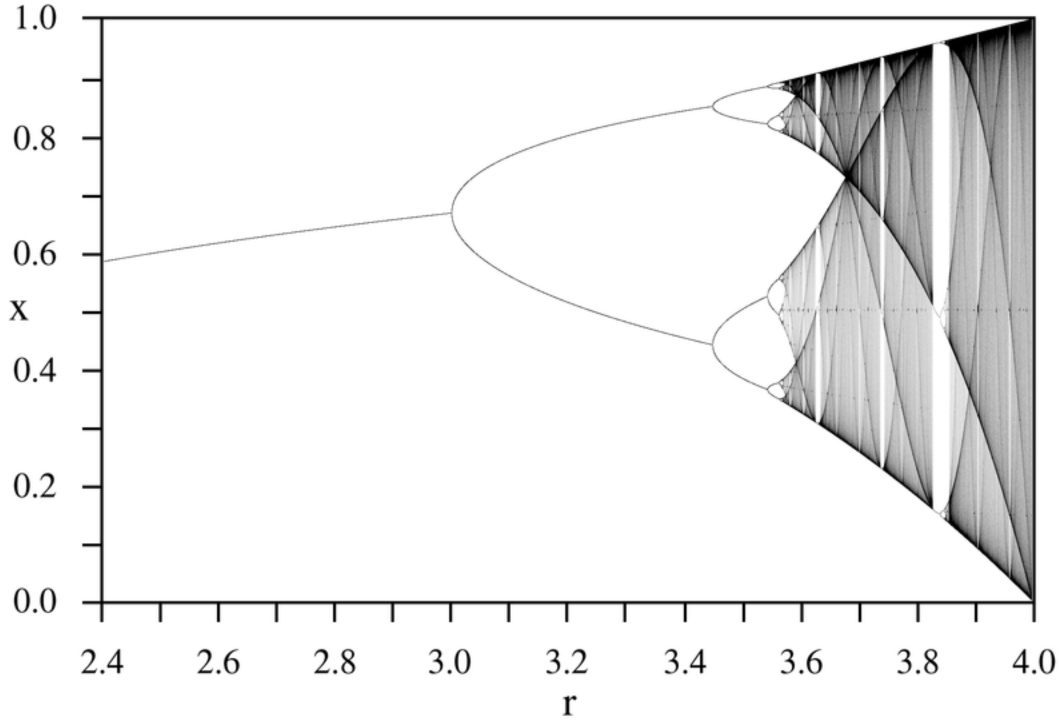
A similar analysis can be performed for other maps:

(1) the logistic map: $f_a : [0,1] \to [0,1]$, $a \in [0,4]$,

$$f_a(x) = ax(1-x).$$

The study of the orbits when a varies leads to a so called "bifurcation diagram" (courtesy of Wikipedia):

Date: Updated on March 1.



You need to create this diagram as part of the project.

(2) The guassian map: $G(x) = e^{-ax^2} + \beta$.

(3) The Henon map $F : \mathbb{R}^2 \to \mathbb{R}^2$

$$F(x, y) = (1 + y - ax^2, bx)$$

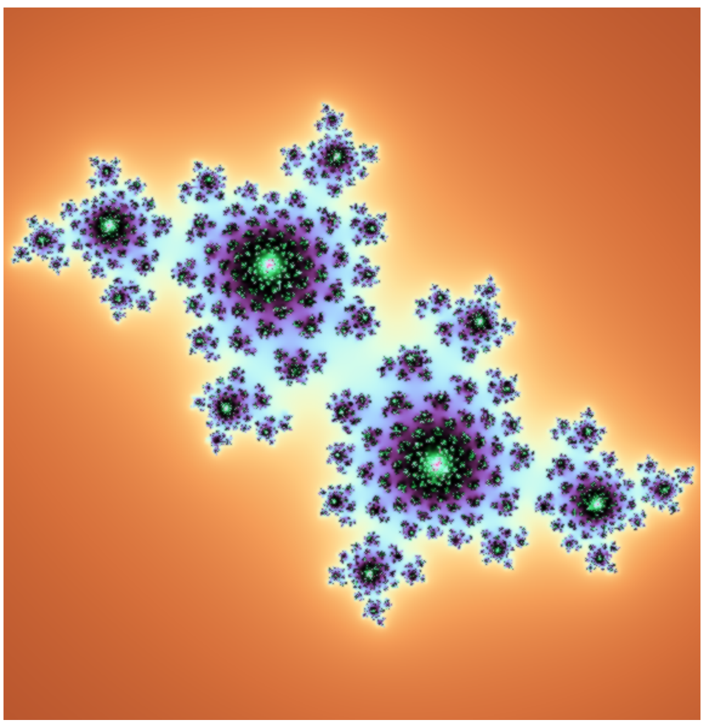
where $a > 0$ and $|b| < 1$. The "atractor" of the Henon map is

2. Julia and Mandelbrot sets

Let $f_c : \mathbb{C} \to \mathbb{C}$ be defined via $f_c(z) = z^2+c$. The *Julia set* $J(c)$ is the set of points that lie at the boundary between the points whose orbit under f are bounded and those points whose orbit under f are undbounded. Prove the following properties of the Julia set (some are harder than the other; you don't have to prove all of them):

(1) The set $J(c)$ is a repeller.
(2) The set $J(c)$ is an invariant ($f(J(c)) = J(c)$ and $f^{-1}(J(c)) = J(c)$)). Use this property to build an algorithm to draw Julia sets on the computer.
(3) An orbit on J is either periodic or chaotic.
(4) All unstable periodic points of f are on $J(c)$.
(5) If one point z_n in the orbit of z_0 has the property that $|z_n| > \max\{|c|, 2\}$ then z_0 escapes. Use this property to construct an algorithm to draw Julia sets on the computer.
(6) The set $J(c)$ is either connected or totally disconnected.
(7) The set $J(c)$ is connected iff the orbit generated by 0 under the map f_c is bounded.
(8) The set $\{c \in \mathbb{C} : J(c) \text{ is connected}\}$is called the Mandelbrot set. Draw this set on the computer.

Here is a picture of a Julia set (I forgot for what value of c). It is the same picture as on the course websites.



3. Random Walks

Let $\{S_n\}$ be a random walk in $\mathbb{R}^n$, $n = 1, 2, 3$ (as defined in class).

(1) What is the probability of return in $2m$ steps? (We will do $n = 1$ in class)
(2) What is the probability of the first return in $2m$ steps? ($n = 1$ in class)
(3) What is the probability of eventual return? That is, what is the probability that one will come back to the origin eventually?

(4) What is the probability of no equalizations in a walk of length $2m$?

(5) What if (for $n = 1$) the distribution function is

$$f_X(x) = \begin{cases} p & \text{if } x = 1 \\ q & \text{if } x = -1 \end{cases},$$

where $0 \leq p, q \leq 1$ and $p + q = 1$. This problem is called the Gambler's ruin. Why?

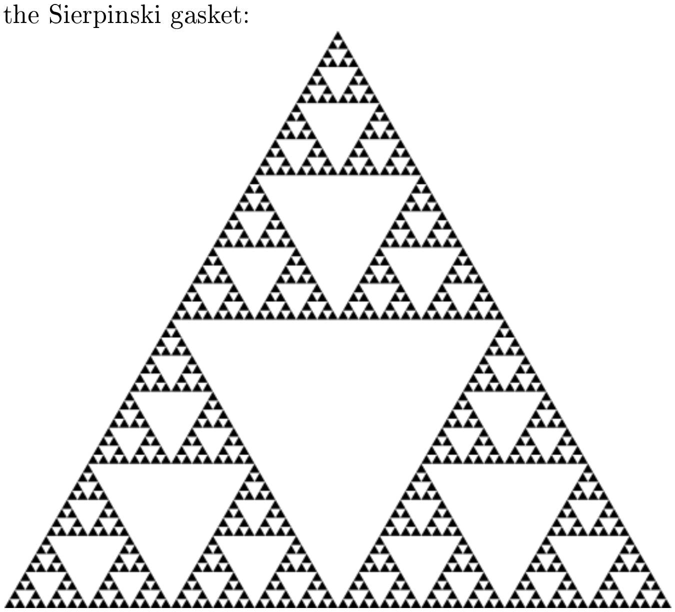
4. Iterated Function Systems

An iterated function system (i.f.s.) on $\mathbb{R}^n$ (you can assume, for simplicty, that $n = 1$ in the proofs) is a collection of contraction maps $(f_1, f_2, \ldots, f_N)$. We let $\mathcal{K}(\mathbb{R}^n)$ be the set of nonempty compact subset of $\mathbb{R}^n$ and let D be the "Hausdorff metric" on $\mathcal{K}(\mathbb{R}^n)$.

(1) Let $F(A) = f_1(A) \bigcup \cdots \bigcup f_n(A)$ for all $A \in \mathcal{K}(\mathbb{R}^n)$. Show that F is a contraction on $\mathcal{K}(\mathbb{R}^n)$. This implies that there is a unique compact subset K of $\mathbb{R}^n$ such that

$$f_1(K) \bigcup \ldots f_n(K) = K.$$

We say that K is self-similar. In general K is a fractal. Here is a picture of the Sierpinski gasket:



(2) Let $k_1, k_2, k_3, \ldots$ be an infinite sequence in the set $\{1, 2, \ldots, N\}]$. Let $a \in \mathbb{R}^2$. Let the sequence $\{x_n\}$ be defined by

$$x_0 = a;\ x_n = f_{k_n}(x_{n-1}) \text{ for } n \geq 1.$$

Then every cluster point of the sequence $\{x_n\}$ belongs to K; every point on K is a cluster point of such a sequence $\{x_n\}$ fir some choice of k_i. Moreover, there is a point a and choice sequence k_i so that K is exactly equal to the set of all cluster points of $\{x_n\}$.

(3) A point x in K is periodic for the i.f.s. if there is a *finite* sequence $k_1, \dots, k_n$ such that
$$x = f_{k_1} \circ f_{k_2} \circ \cdots \circ f_{k_n}(x).$$
Prove that K is the closure of the periodic points of the i.f.s.

(4) If $f_i(K) \bigcap f_j(K) = \emptyset$ then we call the i.f.s. totally disconnected. For example, the i.f.s. that generates the Cantor set is totally disconnected. If the i.f.s. is totally disconnected then one can define a dynamical system $T : K \to K$ with the property that $T \circ f_i(x) = x$ for all $x \in K$. Prove that T is a *chaotic* dynamical system (see the first project).

(5) If the i.f.s. is totally disconnected then K is totally disconnected. Otherwise K is connected.

(6) Suppose now that the maps f_i depend also on a parameter $p \in [0, 1]$. That is, each map is defined now $f_i : [0, 1] \times \mathbb{R}^n \to \mathbb{R}^n$ such that $f_i(p, \cdot)$ is a contraction for all $p \in [0, 1]$. Fixing a given p we have just a regular i.f.s. Let $K(p)$ be its self-similar set. Prove that the map $p \in [0, 1] \mapsto K(p) \in \mathcal{K}(\mathbb{R}^n)$ is a continuous map with respect to the Hausdorff metric. That is, prove that for any $\varepsilon > 0$ there is $\delta > 0$ such that if $|p_1 - p_2| < \delta$ then $D(K(p_1), K(p_2)) < \varepsilon$.

5. Energy and Laplacian on P.C.F. Fractals

The P.C.F. fractals are defined in the fourth chapter of the notes that I photocopied for you. You can see a few examples of them on page 92 (Example 4.1.1 through Example 4.1.5). The goal of this project is to redo explicitly the computations from Sections 1.3, 3.2 and 3.3 for one of these examples. Each fractal counts as a separate project. In addition, you should use your computations and algorithms to study numerically the harmonic functions, eigenvalues, and eigenfunctions on fractals. That is, you should write programs which graph harmonic functions on a fractal based on the initial values (the initial values can be defined only on some n-cell, by Exercise 1.3.1), describe a list of the first 100 eigevalues (maybe less or maybe more, depending on how good your program is), as well as graphing approximations of eigenfunctions corresponding to a few of those eigenvalues. Do you notice any "gap" in the quotient λ_i/λ_j, where λ_i and λ_j are distinct eigenvalues of $-\Delta$?

6. Map Coloring

A *map* is a finite graph on the sphere such that no face lies on two sides of an edge. A *xoloring* of the map connsists in assigning a color to each face (country) so that no two faces that share an edge (boundary) have the same color. Questions:

- When can a map be colored using only two colors?
- Prove that every map can be colored with 5 or fewer colors. (Actually, there is a theorem which says that every map can be colored with 4 or fewer colors; this result is, however, much harder to prove).

7. The general linear group

Let M_n be the set of invertible matrices and define GL_n the set of *invertible* matrices in M_n. GL_n is called the general linear group. Prove the following properties of GL_n:

- GL_n is an open subset of M_n.
- The function $A \mapsto A^{-1}$ is continuous on GL_n.
- GL_n is dense in M_n.
- $GL_n(\mathbb{C})$ is arcwise connected.
- $GL_n(\mathbb{R})$ has two components:
$$\{T \in GL_n(\mathbb{R}) \,:\, \det T > 0\}$$
and
$$\{T \in GL_n(\mathbb{R}) \,:\, \det T < 0\}.$$

8. Fourier Series

In this project you are asked to prove a few results about the convergence of Fourier series. Recall that for a 2π-periodic integrable function f the n-th Fourier coefficient is
$$\widehat{f}(n) = \frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx,$$
for all $n \in \mathbb{Z}$. Define the partial sum
$$S_N(f)(x) = \sum_{n=-N}^{N} \widehat{f}(n)e^{inx}.$$
The big question is: does $S_N(f)(x)$ converges to $f(x)$? The short answer is that not always! Even if f is continuous. Prove, however, the following results (we assume always that the function f is Riemann integrable):

- (uniqueness of the Fourier series) If $\widehat{f}(n) = 0$ for all n and f is continuous at x then $f(x) = 0$.
- If the Fourier series is absolutely convergent then $S_N(f)(x)$ converges *uniformly* at $f(x)$.
- Prove that $S_N(f)(x)$ is *Cesaro summable* to $f(x)$ whenever x is a continuity point of f. Moreover, if f is continuous on $[0.2\pi]$ then the Fourier series is uniformly Cesaro summable to $f(x)$.
- Prove that
$$\frac{1}{2\pi}\int_0^{2\pi} |f(x) - S_N(f)(x)|^2 dx \to 0$$
as $N \to \infty$ (mean-square convergence).
- If f is differentiable at x_0 then $S_N(f)(x_0) \to f(x_0)$ as $N \to \infty$.
- Finally, show that there is a continuous function whose Fourier series diverges at a point.